Inspired by the real hyperbolic case, one can attempt to do the following: Take an imaginary quadratic field E and consider $\mathrm{SU}(n,1;\mathcal{O}_E)$. Borel–Harish-Chandra says that this is a lattice in $\mathrm{SU}(n,1)$ and Godemont says that this is non-cocompact. We attempt to make a more systematic study of lattices in $\mathrm{SU}(n,1)$, reminding the reader that we are viewing this as a *real* algebraic group, although it carries a natural complex structure.

1 Hermitian matrices

We denote the set of Hermitian matrices by $\mathcal{H}$ and the set of invertible Hermitian matrices by $\mathcal{H}^\times$. Given $H \in \mathcal{H}^\times$ there is an associated non-degenerate Hermitian form $\langle \cdot, \cdot \rangle_H$ given by $\langle x, y \rangle_H = y^* H x$. The $H-$unitary group $\mathrm{U}(H)$ is the subgroup of complex matrices which fix the associated form and $\mathrm{SU}(H)$ consists of the matrices with determinant 1.

Definition. $H \in \mathcal{H}^\times$ is defined over a subfield $k \subset \mathbb{C}$ if there is a basis such that in this basis H has coefficients in k. This makes $\mathrm{SU}(H)$ a k-defined real algebraic group.

Two Hermitian matrices are said to be *isometric* if the associated Hermitian pairs on $\mathbb{C}^n$ are isometric. More generally, two k-defined Hermitian matrices H_1, H_2 are equivalent if there exists $\alpha \in k^\times$ such that αH_1 and H_2 are isometric.

The importance is the following:

Theorem 1. *Two k-defined Hermitian matrices H_1, H_2 are equivalent over k if and only if $\mathrm{SU}(H_1) \cong \mathrm{SU}(H_2)$ as real $k \cap \mathbb{R}$-defined algebraic groups.*

Recall further that a matrix is isotropic if there exists a non-zero v such that $\langle v, v \rangle_H = 0$ and anisotropic otherwise. Any $\mathbb{C}^n$ has a decomposition $\mathbb{C}^n = V_{iso} \oplus V_{an}$ into a purely isotropic and anisotropic parts.

Theorem 2. *If H is isotropic, then $\mathrm{SU}(H;k)$ contains a non-trivial unipotent.*

2 Arithmetic lattices of first type

Our first, and simplest, construction resembles the real hyperbolic case.

Let E/F be a totally imaginary quadratic extension of a totally real number field F of degree s over $\mathbb{Q}$. Fix a complex embedding τ_1 of E and a compatible real embedding σ_1 of F. Using these embeddings, we can identify $E \subset \mathbb{C}$ and $F = E \cap \mathbb{R}$. Given an E-defined Hermitian matrix H of signature $(n,1)$, we have the associated group $\mathrm{SU}(H)$. For each embedding $\tau_j \neq \tau_1$ and compatible real embedding $\sigma_j \neq \sigma_1$, we obtain a new Hermitian matrix by applying τ_j to the coefficients of H. We denote the resulting Hermitian matrix by ${}^{\tau_j}H$ and new associated group $\mathrm{SU}({}^{\tau_j}H)$ by ${}^{\tau_j}\mathrm{SU}(H)$. In total, we obtain an injection

$$\mathrm{SU}(H) \longrightarrow \prod_{j=1}^{s} {}^{\tau_j}\mathrm{SU}(H)$$

via the diagonal embedding. The latter group is nothing more than $\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{SU}(H))$, which is $\mathbb{Q}$-defined. By construction, $\mathrm{SU}(H;\mathcal{O}_E)$ and $\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{SU}(H))(\mathbb{Z})$ are commensurable. Finally, observe that we have a natural projection

$$\pi : \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{SU}(H)) \longrightarrow \mathrm{SU}(H)$$

with kernel

$$\ker \pi = \prod_{j=2}^{s} {}^{\tau_j}\mathrm{SU}(H).$$

Under the mapping π, $\pi(\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{SU}(H))(\mathbb{Z}))$ and $\mathrm{SU}(H; \mathcal{O}_E)$ are commensurable. To obtain lattices in $\mathrm{SU}(n,1)$, we can use the diagram

$$\begin{tikzcd}
 & & & 1 \arrow[d] & \\
1 \arrow[r] & \prod_{j=2}^{s} {}^{\tau_j}\mathrm{SU}(H) \arrow[r] & \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{SU}(H)) \arrow[r] & \mathrm{SU}(H) \arrow[r] \arrow[d] & 1 \\
 & & & \mathrm{SU}(n,1) \arrow[d] & \\
 & & & 1 &
\end{tikzcd}$$

We will use the following handy lemma:

Lemma 3. *Let X, Y, and Z be locally compact topological groups, $\Lambda < Z$ be a lattice, and*

$$1 \longrightarrow Y \xrightarrow{t} Z \xrightarrow{\pi} X \longrightarrow 1,$$

a split exact sequence. Then $\pi(\Lambda) = \Gamma$ is a lattice in X if and only if Y is compact. In addition, if Y is compact, then Λ is cocompact if and only if Γ is cocompact.

We are now able to prove the following:

Theorem 4. $\mathrm{SU}(H; \mathcal{O}_E)$ *is a lattice in* $\mathrm{SU}(H)$ *if and only if* ${}^{\tau_j}\mathrm{SU}(H)$ *is compact for all $j = 2, \ldots, s$.*

Proof. From our discussion above, we have the short exact sequence

$$1 \longrightarrow \prod_{j=2}^{s} {}^{\tau_j}\mathrm{SU}(H) \xrightarrow{t} \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{SU}(H)) \xrightarrow{\pi} \mathrm{SU}(H) \longrightarrow 1$$

and this has been constructed so that $\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{SU}(H))$ is a $\mathbb{Q}$-real algebraic group and $\pi(\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{SU}(H))(\mathbb{Z}))$ and $\mathrm{SU}(H; \mathcal{O}_E)$ are commensurable. By Borel–Harish-Chandra, the group $\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{SU}(H))(\mathbb{Z})$ is a lattice in $\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{SU}(H))$. The theorem now follows from an application of the previous lemma. $\square$

The reader can see that this construction is a generalisation of the construction of the lattices $\mathrm{SU}(n, 1; \mathcal{O}_E)$, where E is an imaginary quadratic number field. In this case, since E has only one complex embedding, up to complex conjugation, the compactness condition was vacuously satisfied.

Definition. A pair $(H, E/F)$ is said to be *admissible* if ${}^{\tau_j}\mathrm{SU}(H)$ is compact for all $j = 2, \dots, s$.

Corollary 5. *If $(H, E/F)$ is admissible, then $\mathrm{SU}(H; \mathcal{O}_E)$ is non-cocompact if and only if E is an imaginary quadratic number field.*

This is because a number field of degree n always has exactly n embeddings into an algebraically closed field of characteristic 0 (by the primitive element theorem), so there is always another embedding when F isn't $\mathbb{Q}$.

Proof. Since any unipotent in $\mathrm{SU}(H; \mathcal{O}_E)$ produces a unipotent in the group $\ker \pi$, it is enough to prove that compact algebraic groups cannot possess any nontrivial unipotent elements.

For a compact algebraic group $K < \mathrm{GL}(n; \mathbb{C})$ with associated Lie algebra $\mathfrak{k}$ and a nontrivial unipotent element $x \in K$, we can conjugate K in $\mathrm{GL}(n; \mathbb{C})$ such that $g^{-1}xg$ is upper triangular with ones along the diagonal. For $g^{-1}xg$, there exists $X \in \mathfrak{k}$ such that $\exp(X) = x$. In fact, $\mathbb{R}X$ maps embeds into K which violates the compactness of K. The proof is completed by applying Godemont. □

Any lattice $\Lambda < \mathrm{SU}(n, 1)$ which is commensurable in the wide sense with a lattice of the form $\mathrm{SU}(H; \mathcal{O}_E)$ for an admissible pair $(H, E/F)$ is called an arithmetic lattice of first or simplest type.

We will want to know when these are different. Over a number field it turns out that this is controlled by the underlying Hermitian form, so we turn next to the classification. We will not pursue this further, but there exist infinitely many commensurability classes of lattices of the first type distinguished by the invariant trace field.

2.1 Classification of Hermitian forms

Let V be a finite dimensional E/F vector space equipped with a Hermitian pairing H. We denote the associated $\mathbb{C}$-vector space by $V_{\mathbb{C}}$ and the Hermitian pairing extended to $V_{\mathbb{C}}$ by H. Above, we associated to H a signature pair (p, q). A related invariant on the form H is the number $\sigma_H = |p - q|$. For two different pairings H_1 and H_2, we have the following.

Theorem 6. *As real algebraic groups $\mathrm{SU}(H_1)$ and $\mathrm{SU}(H_2)$ are real isomorphic if and only if $\sigma_{H_1} = \sigma_{H_2}$.*

For an E-defined form H on $V_{\mathbb{C}}$, we have the pair $(\dim_E V, \sigma_H)$.

For a different embedding τ_j of E into $\mathbb{C}$, we obtain a new signature for any Hermitian form H on V. If V_∞ denote the set of inequivalent complex embeddings of E, for each $\nu \in V_\infty$, we obtain a signature $\sigma_H(\nu)$ called the ν-signature.

For a Hermitian form H on $V_{\mathbb{C}}$, by selecting an E-basis for V, we can associate to H a matrix $T_{H,\mathcal{B}}$. The determinant of this matrix is not independent of the selection of the basis $\mathcal{B}$ but is independent viewed as an element of $F^\times / N_{E/F}(E^\times)$. This produces an invariant $\det H \in F^\times / N_{E/F}(E^\times)$ which we call the *determinant of* H and denote by $\det H$.

Theorem 7 (Classification of forms)**.** *Two forms H_1 and H_2 on V and defined over E/F are equivalent if and only if $\sigma_{H_1}(\nu) = \sigma_{H_2}(\nu)$ for all $\nu \in V_\infty$ and* $\det(H_1) = \det(H_2)$.

As a consequence of this result, if we fix the signatures at each $\nu \in V_\infty$, we see that there are precisely two classes of Hermitian forms.

For any form H over E/F on a $\mathbb{C}$-vector space V, we form the invariant tuple $(\dim V, \sigma_\nu(H), \det H)$ which takes values in $\mathbb{N} \times \mathbb{N}^{V_\infty(F)} \times F^\times / N_{E/F}(E^\times)$. If we denote this tuple by $\mathrm{Inv}(H)$, the classification theorem states that H_1 and H_2 are equivalent over E if and only if $\mathrm{Inv}(H_1) = \mathrm{Inv}(H_2)$.

2.2 Examples of lattices

For $E = \mathbb{Q}(\sqrt{-d})$, there are two classes of Hermitian forms over E of signature $(n, 1)$ (or equivalently, $(1, n)$). As before, we have a pair of Hermitian forms

$$H_{n+1,0,-1} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & -1 \end{pmatrix}$$

and

$$H_{n+1,0,(-1)^n} = \begin{pmatrix} -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

For $d \neq 1$ and n even, the forms $H_{n+1,0,-1}$ and $H_{n+1,0,1}$ represent the two possible classes. For $E = \mathbb{Q}(\sqrt{-1})$, the remaining exceptional case can be handled by taking $H_{n+1,0,(-1)^n}$ and $H_{n+1,0,-3}$, which works since $-3 \notin N_{\mathbb{Q}(i)/\mathbb{Q}}(\mathbb{Q}(i)^\times)$.

Theorem 8. *Let E be an imaginary quadratic number field, H a Hermitian form over E of signature $(n, 1)$ with n even. Then* $\mathrm{SU}(H)$ *and* $\mathrm{SU}(n, 1)$ *are isomorphic as real $\mathbb{Q}$-algebraic groups.*

Let E/F be a totally imaginary quadratic extension of a totally real number field with a fixed compatible embeddings $\sigma_1 : F \longrightarrow \mathbb{R}$ and $\tau_1 : E \longrightarrow \mathbb{R}$. As always, we now view $\sigma_1 = \mathrm{id}_F$ and $\tau_1 = \mathrm{id}_E$. Let H be a Hermitian form such that $\sigma_{\sigma_1}(H) = n - 1$ and for all $\sigma_j \neq \sigma_1$, $\sigma_{\sigma_j}(H) = n + 1$. According to the classification of forms, there are two equivalence classes of Hermitian forms on E/F with this prescribed signature.

When n is odd, however, one can see that the above matrices define the same form, and this is morally the reason for the following:

Theorem 9 (Parity theorem)**.** *(a) If n is even, there is precisely one wide commensurability class for arithmetic lattices of first type defined over E/F.*

(b) *If n is odd, there are precisely two wide commensurability classes of arithmetic lattices of first type defined over E/F.*

As in the motivating example, many of the lattices one gets this way are non-uniform.

Theorem 10. *Let E be an imaginary quadratic number field, H an E-defined Hermitian form of signature $(n, 1)$ over E. Then H is isotropic over E.*

There are interesting things to be said here:

Theorem 11 (Chinburg-Stover)**.** *Let $\Gamma < \mathrm{SU}(2, 1)$ be an arithmetic lattice of simple type with associated field extension E/F. Then, there is a one-to-one correspondence between*

1. *commensurability classes of arithmetic Fuchsian subgroups of Γ corresponding to totally geodesic projective algebraic curves on $\Gamma\backslash\mathbf{H}^2_{\mathbb{C}}$ and*
2. *$\mathrm{Aut}(F/\mathbb{Q})$-isomorphism classes of F-quaternion algebras A ramified at all but one infinite place of F and at any finite set of nonarchimedean places that do not split in E/F.*

In particular, any such Γ contains infinitely many distinct commensurability classes of such arithmetic Fuchsian subgroups.

This theorem applies to (manifolds commensurable to) Picard modular surfaces, which are one class of Shimura surfaces (Chinburg-Stover also treat the other class: quotients of $\mathbb{H} \times \mathbb{H}$). Since there are infinitely many commensurability classes of properly immersed (necessarily arithmetic) totally geodesic "curves" (i.e. surfaces), and there is only one arithmetic hyperbolic manifold of real dimension 2 up to commensurability, we get in particular there are infinitely many commensurability classes of infinitely many compact curves.

Lattices of first type will be cocompact if and only if $F \neq \mathbb{Q}$, however, and already these examples are of significant interest. We list two recent ones.

Theorem 12 (Llosa-Isenrich–Py)**.** *For all $n \geq 2$ and cocompact arithmetic lattices $\Gamma \leq SU(n, 1)$ of first type, there exists a finite-index subgroup $\Gamma^0 \leq \Gamma$ such that every finite-index subgroup of Γ_0 surjects onto $\mathbb{Z}$ with a kernel of type F_{n-1} but not of type $FP_n(\mathbb{Q})$.*

This answered a question of Brady about finiteness properties of subgroups of hyperbolic groups. The finiteness is proved by complex Morse theory and the not finiteness can be proved by L^2-Betti number considerations. We discuss the proof later, mostly following the original paper.

One of the most notorious open problems in geometric group theory is whether all hyperbolic groups are residually finite. Deligne constructed non-residually finite central extensions of $\mathrm{Sp}_n(\mathbb{Z})$, so one might naturally wonder whether this works for lattices in other Lie groups.

Theorem 13 (Stover-Toledo)**.** *For all $n \geq 2$ and cocompact arithmetic lattices $\Gamma \leq SU(n, 1)$ of first type, the preimage of Γ in any connected cover of $SU(n, 1)$ is residually finite.*

These are branched covers of complex hyperbolic manifolds, and it turns out that these are projective varieties which aren't locally symmetric, answering an old question of Gromov that asks whether such things with hyperbolic fundamental group exist.

Given all the interest in cocompact lattices, let's see how to produce more uniform ones. As in the real hyperbolic case, the more complicated constructions that follow will only produce uniform ones. We recall some concepts from algebra.

3 Central simple algebras

Throughout, E will denote a number field. The reader should keep in mind that E will be a totally imaginary quadratic extension of a totally real number field F for our applications.

By an E-algebra A we mean an E-vector space equipped with an associative multiplicative structure. We say that an E-algebra is *simple* if there are no nontrivial two-sided ideals in A. We say that A is a *central E-algebra* if the center of A, denoted by $Z(A)$, is E. Finally, if every element of $A \setminus \{0\}$ is invertible in A, we call A a *division algebra*.

Example 14.

- E is a central, simple E-algebra.
- $\mathrm{M}(n; E)$ is a central, simple E-algebra.
- Let A be a 4-dimensional E-vector space with the basis $1, x, y, z$ such that

  $$x^2 = a, \quad y^2 = b, \quad xy = z, \quad yx = -xy.$$

  The algebra A is a simple, central E-algebra and is called a *quaternion algebra*, because the quaternions arise from the reals via this construction. Associated to A is the so-called *Hilbert symbol*

  $$\left(\frac{a, b}{E}\right)$$

  which encodes all the data necessary to retrieve the algebra.
- For any central E-algebra A, $\mathrm{M}(r; A)$ is a central E-algebra.

We have the following fundamental 'rigidity' theorem for simple algebras

Theorem 15 (Skolem-Noether theorem)**.** *Let A be a simple k-algebra. Then* $\mathrm{Aut}_k(A) = \mathrm{Inn}(A)$.

Corollary 16. *Let A be a simple k-algebra with simple subalgebra B. Then every automorphism of B extends uniquely to an automorphism of A.*

Rigidity implies hope for a classification, and indeed

Theorem 17 (Wedderburn structure theorem)**.** *Let A be a simple, central finite dimensional E-algebra. Then there exists a central E-division algebra D and an integer r such that $A \cong \mathrm{M}(r; D)$, as E-algebras.*

3.1 Cyclic algebras

Our primary interest is in a special type of algebra known as a cyclic algebra. Given a cyclic extension L/E of degree d with Galois group $\mathrm{Gal}(L/E) = \langle\theta\rangle$ and $\alpha \in E^\times$, we define a central simple E-algebra $(L/E, \theta, \alpha)$ by:

$$(L/E, \theta, \alpha) = \left\{ \sum_{j=0}^{d-1} \beta_j X^j : \beta_j \in L \right\}$$

subject to the relations

$$X^d = \alpha, \quad X\beta = \theta(\beta)X, \quad \beta \in L.$$

By bash, one shows

Proposition 18. *$(L/E, \theta, \alpha)$ is a central simple E-algebra of dimension d^2 over E as a vector space.*

The norm of an element of L over E is defined by

$$N_{L/E}(\beta) = \prod_{j=0}^{d-1} \theta^j(\beta).$$

As in the real hyperbolic case, we will later need division algebras, so here's a convenient criterion that shows certain cyclic algebras are division algebras.

Theorem 19 (Wedderburn)**.** *(a) If $\alpha^j \notin N_{L/E}(L^\times)$ for $j = 1, \ldots, d-1$, then $(L/E, \theta, \alpha)$ is a division algebra. In addition, if d is prime, this is necessary as well.*

(b) $\alpha \in N_{L/E}(L^\times)$ if and only if $(L/E, \theta, \alpha) \cong \mathrm{M}(d; E)$.

3.2 Splitting fields and degrees

In the case of the reals, we either get quaternions or $M_2(\mathbb{R})$. The latter case is built out of $\mathbb{R}$ very simply, so it is said to split. Analogously, we say that an extension K of E *splits* A if $A \otimes_E K = \mathrm{M}(m; K)$. We review here some useful facts about splitting, though not all of them will be used in the sequel.

The next proposition allows us to view the cyclic algebra $A = (L/E, \theta, \alpha)$ inside $\mathrm{M}(d; L)$.

Proposition 20. *$A \otimes_E L = \mathrm{M}(d; L)$ as central simple L-algebras.*

One can write down the isomorphism by staring very hard at the objects. Up to enlarging the field, everything splits, much like polynomials over fields.

Proposition 21. *Every finite dimensional E-algebra splits over a finite extension of E.*

Definition. The *degree* of A to be the minimum degree over E of a splitting field for A and denote this by $\deg_E(A)$.

Proposition 22. *If $A = (L/E, \theta, \alpha)$ is a division algebra, then $\deg_E(A) = [L : E]$. More generally, $\deg_E(A) \mid [L : E]$.*

The *index* ind(A) is the unique positive integer such that

$$\mathrm{ind}_E(A)^2 \deg_E(A)^2 = \dim_A A.$$

By Wedderburn's structure theorem, $A = \mathrm{M}(m; D)$, for some central E-division algebra D. Alternatively, the index and degree of A are given by $\mathrm{ind}_E(A) = m$ and $\deg_E(A) = \deg_E(D)$.

Proposition 23. *Let A be a simple E-algebra, L/E a finite extension, and $\deg_A = [L : E]$. Then L splits A if and only if A contains a maximal subfield isomorphic to L.*

Proposition 24. *Let $L \subset A$ be a maximal subfield of a simple E-algebra A. Then A splits over L.*

Orders in algebras

In building lattices in $\mathrm{SU}(n, 1)$ using Hermitian matrices, our lattices were constructed, up to commensurability, as stabilizers of lattices in Hermitian vector spaces. Indeed, for an admissible pair $(E/F, H)$ where H has signature $(n, 1)$, the group $\mathrm{SU}(n, 1; \mathcal{O}_E)$ is precisely the stabilizers of the lattices $\mathcal{O}_E^{n,1}$ in $\mathbb{C}^{n+1}$ under the action of the H-unitary group $\mathrm{SU}(H)$. We now introduce orders in algebras which will play the role of the lattice $\mathcal{O}_E^{n+1}$ in the vector space $\mathbb{C}^{n+1}$.

For an E-algebra A, by an $\mathcal{O}_E$-order in A, we mean a finitely generated subring $\mathcal{O}$ of A such that

(a) $\mathcal{O}$ is a finitely generated $\mathcal{O}_E$-module, and

(b) $A = \mathcal{O} \otimes_{\mathcal{O}_E} E$ as an $\mathcal{O}_E$-module.

Example 25.
- $\mathcal{O}_E$ is a $\mathcal{O}_E$-order in E. For any finite extension L/E, $\mathcal{O}_L$ is a $\mathcal{O}_E$-order in L.
- $\mathrm{M}(d; \mathcal{O}_E)$ is an $\mathcal{O}_E$-order in $\mathrm{M}(d; E)$.
- For $A = \left(\frac{-1,-1}{\mathbb{Q}}\right)$, $\mathbb{Z}[1, i, j, k]$ is a $\mathbb{Z}$-order.
- For $A = (L/E, \theta, \alpha)$, $\mathcal{O} = \bigoplus_{j=0}^{d-1} \mathcal{O}_L X^j$ is a $\mathcal{O}_E$-order.

The important thing for us to know is that they exist.

Theorem 26 (Existence of maximal orders)**.** *Let E be a number field (or local field), A an E-algebra, and $\mathcal{O}$ a $\mathcal{O}_E$-order of A. Then $\mathcal{O}$ is contained in a maximal $\mathcal{O}_E$-order.*

4 Second type lattices

For lattices of first type in $\mathrm{SU}(n, 1)$, when $E/\mathbb{Q}$ was an imaginary quadratic extension, the groups $\mathrm{SU}(n, 1; \mathcal{O}_E)$ were lattices in $\mathrm{SU}(n, 1)$. However, when E was a larger extension of $\mathbb{Q}$, the indiscreteness of $\mathcal{O}_E \subset \mathbb{C}$ complicated the matter considerably. This persists in our constructions of lattices in this section. For this reason, we begin with the case $E/\mathbb{Q}$ is an imaginary quadratic extension and generalize the construction to arbitrary totally imaginary quadratic extensions of totally real number fields for which this construction is a special case.

Definition. A cyclic E-algebra A equipped with a standard involution of second kind is said to be a unitary algebra over E/F.

Let A be a unitary division algebra of degree $n+1$ over an imaginary quadratic extension E of $\mathbb{Q}$. For a Hermitian element $h \in \mathcal{H}^\times$ of signature $(n,1)$, the group $\mathrm{SU}(h)$ is isomorphic to $\mathrm{SU}(n,1)$ via the splitting isomorphism $A \otimes_E \mathbb{C} \cong \mathbb{M}(n+1;\mathbb{C})$. Given an $\mathcal{O}_E$-order $\mathcal{O}$ in A, we form the group

$$\mathrm{SU}(h;\mathcal{O}) \stackrel{\text{def}}{=} \left\{x \in \mathcal{O}^\times : xx^{*h} = 1\right\}.$$

Theorem 27. $\mathrm{SU}(h;\mathcal{O})$ *is a cocompact lattice in* $\mathrm{SU}(h)$.

Proof. That $\mathrm{SU}(h;\mathcal{O})$ is a lattice follows from Borel–Harish-Chandra, since $\mathrm{SU}(h;\mathcal{O})$ can be viewed as the $\mathbb{Z}$-points of the real $\mathbb{Q}$-algebraic group $\mathrm{SU}(h)$. We demonstrate cocompactness by arguing the contrapositive. If $\mathrm{SU}(h;\mathcal{O})$ is not cocompact, by Godemont's criterion, there exists a nontrivial unipotent element $x \in \mathrm{SU}(h;\mathcal{O})$. This, in turn, produces the nontrivial nilpotent element $x-1 \in A$, which contradicts our assumption that A is a division algebra. $\square$

Let E/F be a totally imaginary quadratic extension of a totally real number field F. We denote the distinct complex embeddings of E be $\tau_1, \dots \tau_s$ and the compatible real embeddings of F by $\sigma_1, \dots, \sigma_s$. Via τ_1 and σ_1, we identify $E \subset \mathbb{C}$ and $F \subset \mathbb{R}$ such that $E \cap \mathbb{R} = F$. Given a cyclic extension L/E, for each embedding τ_j of E, we obtain a family of embeddings $\lambda_{1,j}, \dots, \lambda_{r_j,j}$. We select a fixed embedding λ_j for each τ_j and view $L \subset \mathbb{C}$ via λ_1.

Given a cyclic E-algebra $A = (L/E, \theta, \alpha)$ via the embedding $L \subset \mathbb{C}$ afforded by λ_1 and the splitting $A \otimes_E L \cong \mathrm{M}(d;L)$, for each embedding $\lambda_j \neq \lambda_1$, we obtain a new algebra

$${}^{\lambda_j}A = (\lambda_j(L)/\tau_j(E), \lambda_j \circ \theta \circ \lambda_j^{-1}, \tau_j(\alpha)).$$

Up to an E-algebra isomorphism, this algebra is independent of the selection of λ_j among the family $\lambda_{1,j}, \dots, \lambda_{r_j,j}$. We denote this algebra by ${}^{\tau_j}A$.

In addition, if $(A,*)$ is a unitary algebra with Hermitian element $h \in \mathcal{H}^\times$, for each $\tau_j \neq \tau_1$, we obtain a new Lie group ${}^{\tau_j}\mathrm{SU}(h) = \mathrm{SU}({}^{\tau_j}h)$. As before, applying $\mathrm{Res}_{F/Q}$ we obtain the short exact sequence

$$1 \longrightarrow \prod_{j=2}^{s} {}^{\tau_j}\mathrm{SU}(h) \stackrel{t}{\longrightarrow} \mathrm{Res}_{F/Q}(\mathrm{SU}(h)) \stackrel{\pi}{\longrightarrow} \mathrm{SU}(h) \longrightarrow 1.$$

We start with $(A,*)$, a unitary division algebra of degree $n+1$ over E with a Hermitian element h of signature $(n,1)$. Given an $\mathcal{O}_E$-order $\mathcal{O}$ in A, $\mathrm{SU}(h;\mathcal{O})$ maps to a subgroup of $\mathrm{Res}_{F/Q}(\mathrm{SU}(h))$ which is commensurable with $\mathrm{Res}_{F/Q}(\mathrm{SU}(h))(\mathbb{Z})$ and in the projection π, $\mathrm{Res}_{F/Q}(\mathrm{SU}(h))(\mathbb{Z})$ maps to a subgroup which is commensurable with $\mathrm{SU}(h;\mathcal{O})$.

Theorem 28. $\mathrm{SU}(h;\mathcal{O})$ *is a (cocompact) lattice in* $\mathrm{SU}(n,1)$ *if and only if* ${}^{\tau_j}\mathrm{SU}(h)$ *is compact for each* $j = 2, \dots, s$.

We call these lattices *arithmetic lattices of second type.* That $\mathrm{SU}(h;\mathcal{O})$ is a lattice is identical to the proof of theorem 4. For cocompactness, we can argue as in the proof of the corollary to theorem 4.

5 Mixed type lattices

Let $r, d \in \mathbb{N}$, $rd = n+1$, E/F a totally imaginary quadratic extension of a totally real number field, and $(A, *)$, a unitary algebra over E/F of degree d. An involution $*$ on A is a map $A \to A$ such that $(x+y)^* = x^* + y^*, (xy)^* = y^* x^*$. $*$ is said to be of *first kind* if it acts on E by the identity and of *second kind* otherwise. The simple E-algebra $\mathrm{M}(r; A)$ admits an involution of second kind given by $*$-transposition. If L is a splitting field for A, then

$$\mathrm{M}(r; A) \otimes_E L = \mathrm{M}(r; A \otimes_E L) = \mathrm{M}(r; \mathrm{M}(d; L)) = \mathrm{M}(rd; L).$$

We assume that the involution $*$ on $\mathrm{M}(r; A)$ is standard in this splitting, i.e. the involution extended on $A \otimes_E \mathbb{C}$ is complex transposition.

For a Hermitian element $h \in M(r; A)$ with associated twisted involution $*$, set

$$\mathrm{SU}(h; A) = \{x \in M(r; A) : xx^* = I_r\}.$$

If in the splitting $M(r; A) \otimes_E L$, h has signature $(n, 1)$, $\mathrm{SU}(h; A \otimes_E \mathbb{C}) \cong \mathrm{SU}(n, 1)$. Given a triple $(A, *, h)$ above for the pair (r, d), we say that $(A, *, h)$ is *admissible* if for all $\tau_j \neq \mathrm{id}_E$ the group ${}^{\tau_j}\mathrm{SU}(h; A \otimes_E \mathbb{C})$ is compact.

Theorem 29. *Let $(A, *, h)$ be admissible over E/F with associated pair (r, d). Then for any $\mathcal{O}_E$-order $\mathcal{O}$, $\mathrm{SU}(h; \mathcal{O})$ is a lattice in $\mathrm{SU}(n, 1)$.*

We call these *arithmetic lattices of mixed type.* Note that both arithmetic lattices of first and second type are of mixed type.

Proof. This follows from Borel–Harish-Chandra and lemma 3. ☐

We have, at long last, described all arithmetic lattices.

Theorem 30 (Classification of arithmetic lattices)**.** *Let $(A, *, h)$ be an admissible triple over E/F with associated pair (r, d) and $\mathcal{O}$ an $\mathcal{O}_E$-order in A. Then $\mathrm{SU}(h; \mathcal{O})$ is a lattice in $\mathrm{SU}(n, 1)$ via the injection induced by the isomorphism of $\mathrm{SU}(h)$ with $\mathrm{SU}(n, 1)$ given by the splitting isomorphism $\mathrm{M}(r; A) \otimes_E \mathbb{C} \longrightarrow \mathrm{M}(n+1; \mathbb{C})$. Moreover, if $r = 1$ and E is not an imaginary quadratic extension of $\mathbb{Q}$ or $r > 1$, then $\mathrm{SU}(h; \mathcal{O})$ is cocompact.*

Remark 31. By Skolem-Noether, if we change the splitting (and hence the isomorphism between $\mathrm{SU}(h)$ and $\mathrm{SU}(n, 1)$, we obtain a new lattice which is commensurable in the wide sense with the original one. Thus, up to wide commensurability, this is independent of the selection of a splitting isomorphism.

Corollary 32 (Classification of noncocompact arithmetic lattices)**.** *Let $\Lambda < \mathrm{SU}(n, 1)$ be a noncocompact arithmetic lattice. Then there exists an imaginary quadratic extension $E/\mathbb{Q}$ and an E-defined signature $(n, 1)$ Hermitian matrix H such that Λ is commensurable in the wide sense with $\mathrm{SU}(H; \mathcal{O}_E)$.*

In combination with the parity theorem, we have a very explicit description of the noncocompact arithmetic lattices in $\mathrm{SU}(n, 1)$.

6 Examples

6.1 Lattices of type $U^3_{2,0}$

Given an admissible triple $(A, *, h)$ over E/F with associated pair (r, d), the elements of reduced norm one, $(A \otimes_E \mathbb{C})^1$, is an *outer real F-form* of $\mathrm{SL}(n, \mathbb{C})$, which has the associated *Tits symbol* $U^d_{n,r-1}$ to the group and its lattices.

By 2-adic uniformisation, Mumford constructed the first example of a fake projective plane, i.e. a complex algebraic surface that has the same Betti numbers as $\mathbb{CP}^2$ but not homeomorphic to it. By Yau's uniformisation theorem, the manifold M_{Mum} Mumford constructed is a complex hyperbolic 2-manifold of minimal Euler characteristic. By Wang's theorem, M_{Mum} is a minimal volume complex hyperbolic 2-manifold. It is known that there are 50 fake projective planes, and a consequence of the Calabi conjecture is that each of these has to come from a lattice in $SU(2,1)$. Furthermore, they are determined by their fundamental groups by Mostow rigidity. Klingler proved that every fake $\mathbb{CP}^2$ is arithmetic of type $U^3_{2,0}$, and more recently Stover classified these up to profinite equivalence.

Our first concrete example of a lattice of second type is a lattice whose associated arithmetic orbifold is commensurable with Mumford's fake $\mathbb{CP}^2$.

Example 33. Let $E = \mathbb{Q}(\sqrt{-7})$, $F = \mathbb{Q}$, $L = \mathbb{Q}(\zeta_7)$, and $K = \mathbb{Q}(\cos(2\pi/7))$. Set

$$\lambda = \frac{-1 + \sqrt{-7}}{2}, \quad \alpha = \lambda/\bar{\lambda}.$$

In addition, set θ to be the Galois automorphism

$$\theta : L \longrightarrow L, \quad \theta(\zeta_7) = \zeta_7^2.$$

The reader can check (using Wedderburn's theorem 19) that $A = (L/E, \theta, \alpha)$ is a division algebra.

A admits an involution of second type. Concretely, set

$$\begin{aligned} X^* &= \overline{\alpha} X^2, \quad \beta^* = \overline{\beta}, \quad \beta \in L \\ h &= (\lambda - \bar{\lambda}) - \lambda X + \bar{\lambda} X^2 \\ \mathcal{O} &= \mathcal{O}_L \oplus \mathcal{O}_L \bar{\lambda} X \oplus \mathcal{O}_L \bar{\lambda} X^2. \end{aligned}$$

Then $SU(h; \mathcal{O})$ is an arithmetic lattice in $SU(2,1)$.

The commensurability result is due to Kato:

Theorem 34. *Let $M = \mathbb{H}^2_C / SU(h; \mathcal{O})$. Then M_{Mum} and M are commensurable.*

Although they define manifolds of the same dimension, lattices of type $U^3_{2,0}$ and $U^1_{2,2}$ differ greatly. Here is a geometric difference:

Theorem 35.

- *Every lattice of type $U^1_{2,2}$ contains a totally geodesic real hyperbolic 2-orbifold group.*
- *No lattice of type $U^3_{2,0}$ contains a totally geodesic real hyperbolic 2-orbifold group.*

Proof. For (a), let $A = SU(H;\mathcal{O}_E)$ for a pair $(H, E/F)$. As before, we may assume that $H = \mathrm{diag}(\alpha_1, \alpha_2, \alpha_3)$ with $\alpha_1 < 0$. Set $\Gamma = \mathrm{Stab}(C[e_1, e_2])$. By construction, Γ is a lattice in $SU(1,1)$. Alternatively, let Γ be the subgroup of A fixed under $*$. Then Γ is the lattice $SO(H;\mathcal{O}_F)$, which by construction is in $SO(2,1)$. For (b), given a totally geodesic real hyperbolic 2-orbifold group Γ in a lattice A of type $U^3_{2,0}$, we obtain a quaternion subalgebra $D \subset A$, where A is the cyclic degree 3 division algebra used to A. However, $\dim D = 4$ while $\dim A = 9$, which is impossible. $\square$

6.2 Lattices of type $U^2_{3,1}$

Our next class of examples are the first ones of genuine mixed type. The lattices of type $U^2_{3,1}$ are constructed as follows. Let A be a quaternion algebra over E/F equipped with an involution of second kind $*$. Let $\alpha_1, \alpha_2 \in A^\times$ such that $\alpha_i^* = \alpha_j$ and set $h = \mathrm{diag}(\alpha_1, \alpha_2)$. We select α_1, α_2 such that the signature of α_1 and α_2 in the splitting $A \otimes_E \mathbb{C}$ are $(1,1)$ and $(2,0)$, respectively. In addition, for each other embedding $\tau_\ell \neq \mathrm{id}_E$, we insist that α_1 and α_2 both have signature $(2,0)$. It follows that $SU(h; A \otimes_E \mathbb{C})$ is isomorphic to $SU(3,1)$ and by the classification of arithmetic lattices, for any $\mathcal{O}_E$-order $\mathcal{O}$ of A, $SU(h;\mathcal{O})$ is a lattice in $SU(3,1)$.

Example 36. Let D be a quaternion algebra with Hilbert symbol $\left(\frac{a,b}{\mathbb{Q}(i)}\right)$. Select $\alpha_1 \in D^\times$ such that $SU(\alpha_1; D)$ is a Fuchsian group and $\alpha_2 = 1$. Setting $h = \mathrm{diag}(\alpha_1, \alpha_2)$, $SU(h;\mathcal{O})$ is a lattice of type $U^2_{3,1}$ for any $\mathcal{O}_E$-order $\mathcal{O}$ of D.

As in the previous subsection, lattices of type $U^2_{3,1}$ differ from those of type $U^4_{3,0}$ and $U^1_{3,3}$.

Theorem 37.

- *Every lattice of type $U^1_{3,3}$ contains a totally geodesic real hyperbolic s-orbifold group for $s = 2, 3$.*
- *Every lattice of type $U^2_{3,1}$ contains a totally geodesic real hyperbolic 2-orbifold group but not a totally geodesic real hyperbolic 3-orbifold group.*
- *There exist lattices of type $U^4_{3,0}$ which do not contain a totally geodesic real hyperbolic s-orbifold group for $s = 2, 3$.*

Proof. For a lattice A of type $U^1_{3,3}$, there exists a pair E/F and a Hermitian form $H \in M(4;E)$ such that A is commensurable in the wide sense with $SU(H;\mathcal{O}_E)$. As noted above, we can assume that H is diagonal and $H \in M(4;F)$. Consequently, we can form the group $SU(H;\mathcal{O}_F)$. This group is a totally geodesic real hyperbolic 3-orbifold group, which in turn contains a totally geodesic real hyperbolic 2-orbifold group.

For a lattice A of type $U^1_{3,3}$, we simply take the subgroup Γ of $SU(H;\mathcal{O})$ consisting elements of the form $\mathrm{diag}(\beta, 1)$. Visibly, Γ is a lattice in $SU(1,1)$ which in turn produces a lattice in $SO(2,1)$, since $SO(2,1)$ and $SU(1,1)$ are isogenous.

For a lattice A of type $U^4_{3,0}$, select A not containing a $*$-invariant quaternion subalgebra. The presence of such a totally geodesic subgroup in either case would produce such a quaternion subalgebra $B \subset A$. By our selection of A, this is impossible. $\square$

It is a natural question what totally geodesic submanifolds there are in a hyperbolic manifold. In the real hyperbolic case, constructions such as Gromov–Piatetski-Shapiro give rise to non-arithmetic lattices, but this doesn't work here because complex codimension one is real codimension two, and indeed non-arithmetic complex hyperbolic lattices are much harder to come by. In fact, analogously to the real hyperbolic case, when there are lots of totally geodesic submanifolds one can recover/build an arithmetic structure:

Theorem 38 (Bader-Fisher-Miller-Stover)**.** *Any complex hyperbolic manifold of dimension $n \geq 2$ with finite volume, containing infinitely many maximal properly immersed totally geodesic submanifolds of real dimension at least 2, is arithmetic.*

7 The Albanese

We now turn to proving theorem 12. We recall some important concepts from complex geometry.

Let X be a closed Kähler manifold and let $H^0(X, \Omega_X^1)$ be the space of holomorphic 1-forms on X. If $\theta : [0,1] \to X$ is a path in X we denote by $i(\theta) \in (H^0(X, \Omega_X^1))^*$ the linear map on $H^0(X, \Omega_X^1)$ taking a form α to the integral

$$\int_\theta \alpha.$$

Since holomorphic forms on X are closed, this only depends on the homotopy class of θ relative to its endpoints. Thus, one can also define $i(u)$ for $u \in H_1(X, \mathbb{Z})$. The kernel of the map

$$i : H_1(X, \mathbb{Z}) \to (H^0(X, \Omega_X^1))^*$$

is the torsion subgroup of $H_1(X, \mathbb{Z})$ and its image is a lattice in $(H^0(X, \Omega_X^1))^*$. The Albanese torus of X is defined as

$$\mathrm{Alb}(X) := (H^0(X, \Omega_X^1))^* / i(H_1(X, \mathbb{Z})). \tag{7}$$

For a fixed point $x_0 \in X$ we define a map

$$\mathrm{alb}_X : X \to \mathrm{Alb}(X)$$

by setting $\mathrm{alb}_X(x) = i(\theta_x) \mod i(H_1(X, \mathbb{Z}))$, where θ_x is any continuous path going from x_0 to x. This does not depend on the choice of θ_x. The resulting map alb_X is holomorphic. By construction the differential of the map alb_X at a point $x \in X$ is the evaluation map

$$T_x X \to (H^0(X, \Omega_X^1))^*$$
$$v \mapsto (\alpha \mapsto \alpha_x(v)).$$

Hence we have:

Lemma 39. *Let $x \in X$. The linear map $d\,\mathrm{alb}_X(x)$ is injective if and only if the evaluation map*

$$H^0(X, \Omega_X^1) \to (T_x X)^*$$
$$\alpha \mapsto \alpha_x$$

is onto.

We now turn to the study of the Albanese map in the case of a quotient of the unit ball $B \subset \mathbb{C}^m$ by an arithmetic lattice. Let $\Gamma < \mathrm{PU}(m,1)$ be a cocompact torsionfree lattice. We recall that the commensurator of Γ is defined as follows:

$$\mathrm{Comm}(\Gamma) = \{g \in \mathrm{PU}(m,1) \mid \Gamma \cap g\Gamma g^{-1} \text{ has finite index in both } \Gamma \text{ and } g\Gamma g^{-1}\}.$$

Theorem 40 (Eyssidieux)**.** *Assume that Γ is arithmetic and has positive first Betti number. Then there exists a finite index subgroup $\Gamma_0 < \Gamma$ with the property that the Albanese map of X_{Γ_0} is an immersion.*

We fix $\Gamma < \mathrm{PU}(m,1)$ as in theorem 40. Let Ω^1_B be the space of all holomorphic 1-forms on B. For a lattice $\Lambda < \mathrm{PU}(m,1)$, let $\Omega^1_{B,\Lambda} \subset \Omega^1_B$ be the subspace of Λ-invariant forms. We define a subset $\mathcal{L} \subset \Omega^1_B$ as follows. A holomorphic 1-form α on B belongs to $\mathcal{L}$ if and only if there exists a cocompact lattice $\Lambda < \mathrm{PU}(m,1)$ such that Λ is commensurable to Γ and α is invariant under Λ. Consider the linear subspace

$$V_0 \subset \Omega^1_B$$

spanned by $\mathcal{L}$ and let V be the closure of V_0 for the topology of uniform convergence on compact sets. We first observe that the set $\mathcal{L}$ is invariant under the action of $\mathrm{Comm}(\Gamma)$. Indeed, if $\alpha \in \mathcal{L}$ is invariant under a lattice Λ commensurable with Γ, and if $g \in \mathrm{Comm}(\Gamma)$, then $(g^{-1})^*\alpha$ is invariant under the lattice $g\Lambda g^{-1}$ which is still commensurable with Γ. Hence $g(\mathcal{L}) = \mathcal{L}$ and consequently $g(V_0) = V_0$ and $g(V) = V$ for all $g \in \mathrm{Comm}(\Gamma)$. The density of $\mathrm{Comm}(\Gamma)$ in $\mathrm{PU}(m,1)$ then implies that the space V is $\mathrm{PU}(m,1)$-invariant.

Since the intersection of finitely many lattices commensurable with Γ is again a lattice commensurable with Γ, the following lemma is clear.

Lemma 41. *Let $W \subset V_0$ be a finite dimensional vector subspace. Then there exists a lattice $\Lambda < \mathrm{PU}(m,1)$ commensurable with Γ such that $\gamma^*\alpha = \alpha$ for all $\alpha \in W$ and $\gamma \in \Lambda$.*

Lemma 42. *Let $p \in B$. Let $\mathrm{ev}_p : V \to (\mathbb{C}^m)^*$ be the evaluation map at p. Then ev_p is onto.*

Proof. Since V is $\mathrm{PU}(m,1)$-invariant, it is enough to prove the Lemma for $p = o$, the origin of the ball. If the image $\mathrm{ev}_o(V) \subset (\mathbb{C}^m)^*$ is equal to 0, then ev_p would be equal to 0 for each point $p \in B$ and V would be reduced to zero. This is impossible since $b_1(\Gamma) > 0$. Hence the image of ev_o is nonzero. Since $\mathrm{ev}_o(V) \subset (\mathbb{C}^m)^*$ is invariant under the natural action of $\mathrm{U}(m)$, we must have $\mathrm{ev}_o(V) = (\mathbb{C}^m)^*$. This concludes the proof. □

For a lattice $\Lambda < \mathrm{PU}(m,1)$, we now define the following subset of the ball:

$$Z_\Lambda = \{x \in B \mid d\,\mathrm{alb}_{X_\Lambda} \text{ is not injective at } x \bmod \Lambda\}.$$

Proposition 43. *For each compact subset $M \subset B$, there exists a lattice $\Lambda < \mathrm{PU}(m,1)$ commensurable with Γ such that Z_Λ does not intersect M.*

Proof. We make the following observations. Given a point $p \in B$, lemma 42 implies that there exist elements $\alpha_1, \ldots, \alpha_m$ in V such that the evaluations $(\alpha_i(p))_{1 \le i \le m}$ generate $(\mathbb{C}^m)^*$. Since this is an open condition, we can actually assume that $\alpha_1, \dots, \alpha_m$ belong to V_0. Note that the evaluations $\alpha_1(q), \dots, \alpha_m(q)$ will then be linearly independent for q in a neighborhood of p. This implies that for each compact subset $M \subset B$, there exists a finite dimensional subspace $W \subset V_0$ such that $\mathrm{ev}_p(W) = (\mathbb{C}^m)^*$ for each point $p \in M$. According to Lemma 25, there exists a lattice Λ commensurable with Γ such that $W \subset \Omega^1_{B,\Lambda}$. Thanks to lemma 39, this implies that

$$M \cap Z_\Lambda = \emptyset.$$

$\square$

Proof of theorem 40. Pick a compact fundamental domain $K \subset B$ for the action of Γ on B. According to proposition 43, there exists a lattice $\Lambda < \mathrm{PU}(m, 1)$ commensurable with Γ and such that $Z_\Lambda \cap K = \emptyset$. Let $\Gamma_0 \lhd \Gamma$ be a normal finite index subgroup such that $\Gamma_0 \subset \Lambda \cap \Gamma$. Since $\Gamma_0 < \Lambda$ we must have

$$Z_{\Gamma_0} \subset Z_\Lambda.$$

Hence Z_{Γ_0} does not intersect K. But Z_{Γ_0} is Γ-invariant since $\Gamma_0 \lhd \Gamma$. If $x \in Z_{\Gamma_0}$, there exists $\gamma \in \Gamma$ such that $\gamma \cdot x \in K$. But $\gamma \cdot x$ also lies in Z_{Γ_0}. We thus obtain a contradiction. This shows that $Z_{\Gamma_0} = \emptyset$, implying that the Albanese map of X_{Γ_0} is an immersion. $\square$

8 Proof of theorem 12

The theorem we are interested in is about fibring manifolds, which has been of substantial interest. The fibre, as an infinite cyclic cover, is a natural object to consider, and indeed Milnor did some work here. He showed that some homology must be infinite, and in the next subsection we show a result that lets us conclude the infiniteness is where we want.

8.1 Infiniteness properties of cyclic covers

In this subsection we follow Llosa-Isenrich–Martelli–Py to show that the infinite cyclic cover has infinite homology in the middle dimension. The following is a consequence of Poincare-Lefschetz duality.

Lemma 44. *Let $(M, \partial M)$ be an n-dimensional compact oriented manifold with boundary and let $i \in \{1, \dots, n\}$. Then*

$$b_{n-i}(M) \leq b_{i-1}(\partial M) + b_i(M).$$

With this in hand, we can prove the result we need later to show that certain groups aren't of type FP_n.

Proposition 45. *Assume that $b_i(\ker(f))$ is finite for $i < k-1$. Then the group $H_{k-1}(\ker(f))$ is infinite dimensional. In particular, the group $\ker(f)$ is not of type F_k.*

Proof. We write $M_\mathbb{Z} = \mathbb{H}^{2k}/\ker(f)$ and $M_\ell = \mathbb{H}^{2k}/N_\ell$, where N_ℓ is the kernel of the morphism

$$\Gamma \to \mathbb{Z}/\ell\mathbb{Z}$$

obtained by reducing f mod ℓ. Let $T : M_{\mathbb{Z}} \to M_{\mathbb{Z}}$ be a generator of the group of deck transformations of the cyclic covering space $M_{\mathbb{Z}} \to \mathbb{H}^{2k}/\Gamma$. Let $t : H_*(M_{\mathbb{Z}}) \to H_*(M_{\mathbb{Z}})$ be the transformation induced by T on homology. By applying Milnor's long exact sequence to the cyclic covering $M_{\mathbb{Z}} \to M_\ell$, we obtain the following short exact sequence:

$$0 \to H_i(M_{\mathbb{Z}})/(t^\ell - 1)H_i(M_{\mathbb{Z}}) \to H_i(M_\ell) \to \ker(t^\ell - 1 : H_{i-1}(M_{\mathbb{Z}}) \to H_{i-1}(M_{\mathbb{Z}})) \to 0. \tag{2}$$

For $i \leq k-1$ the vector spaces appearing on the left and right in the above short exact sequence have finite dimension bounded by $b_i(M_{\mathbb{Z}})$ and $b_{i-1}(M_{\mathbb{Z}})$ respectively. We thus obtain

$$b_i(M_\ell) \leq b_i(M_{\mathbb{Z}}) + b_{i-1}(M_{\mathbb{Z}})$$

for $i \leq k-1$ and all $\ell \geq 1$.

Since the restriction of f to each of the finitely many cusps of M is nontrivial, the number of boundary tori in M_ℓ is independent of ℓ. Therefore, for all i, the Betti numbers $b_{i-1}(\partial M_\ell)$ are also independent of ℓ. By lemma 44 we have that $b_{2k-i}(M_\ell) \leq b_{i-1}(\partial M_\ell) + b_i(M_\ell)$ for $i \leq k-1$. Thus, we deduce from (3) that all Betti numbers $b_i(M_\ell)$, for $i \in \{0, \ldots, 2k\}$ with $i \neq k$, are uniformly bounded above (independently of ℓ).

Since the Euler characteristic of M_ℓ is equal to ℓ times that of M (which is nonzero basically by Chern-Gauss-Bonnet), we obtain that $b_k(M_\ell)$ grows roughly linearly with ℓ. In particular

$$b_k(M_\ell) \to \infty$$

as ℓ goes to ∞. Combining Equation (2), this time for $i = k$ with the fact that $b_{k-1}(M_{\mathbb{Z}})$ is finite, we deduce that the codimension of the image of the natural map

$$H_k(M_{\mathbb{Z}}) \to H_k(M_\ell)$$

is bounded above uniformly in ℓ. This implies that $H_k(M_{\mathbb{Z}})$ is infinite dimensional and completes the proof. □

8.2 Back to the proof

The key intermediate step is

Theorem 46. *Let X be a closed aspherical Kähler manifold of complex dimension $m \geq 2$.*

1. *Let β be a holomorphic 1-form on X with finitely many zeros. Then the cohomology class $b = [\mathrm{Re}(\beta)] \in H^1(X, \mathbb{R}) \simeq H^1(\pi_1(X), \mathbb{R})$ lies in $\Sigma^{m-1}(\pi_1(X)) \cap -\Sigma^{m-1}(\pi_1(X))$. If b is rational then its kernel is of type $\mathcal{F}_{m-1}$; if furthermore the Euler characteristic of X is nonzero, the kernel of b is not of type* $\mathrm{FP}_m(\mathbb{Q})$.

2. *Let $\psi : X \to A$ be a holomorphic map to a complex torus. Assume that ψ is a finite map. Let α be a holomorphic 1-form on A which does not vanish on any nontrivial subtorus of A. Then the form $\psi^*\alpha$ has finitely many zeros. Consequently, the class $[\mathrm{Re}(\psi^*\alpha)]$ lies in $\Sigma^{m-1}(\pi_1(X)) \cap -\Sigma^{m-1}(\pi_1(X))$.*

We recall that a map $f : X \to Y$ between two topological spaces is said to be finite if each of its fibers $f^{-1}(y)$ $(y \in Y)$ is a finite set. The second item reduces to the first by

Proposition 47. *Under the assumptions of theorem 46, the form $\psi^*\alpha$ has only finitely many zeros on X.*

Proof sketch. Let Z be a connected component of the set of zeros of $\psi^*\alpha$. If $\psi(Z)$ is positive dimensional, it generates a nontrivial subtorus of A on which α vanishes. This is a contradiction. Hence $\psi(Z)$ must be zero dimensional and thus a point. Since ψ is finite, this implies that Z is a point. □

The following result of Simpson, the complex analogue of a standard result in real Morse theory, is the main tool need to show the first item.

Theorem 48. *Let Y be a compact Kähler manifold of complex dimension $m \geq 2$ and β be a holomorphic 1-form on Y with finitely many zeros. Let $\hat{Y}$ be the universal cover of Y and $f : \hat{Y} \to \mathbb{R}$ be a primitive of the lift to $\hat{Y}$ of the form* $\mathrm{Re}(\beta)$*. Then for all real numbers c, d such that $c \leq d$ the inclusion*

$$f^{-1}([d, \infty)) \subset f^{-1}([c, \infty))$$

induces an isomorphism on π_i for $i \leq m - 2$ and a surjection on π_{m-1}.

Proof of theorem 46. Since β and $-\beta$ both have finitely many zeros, it is enough to show that $b = [\mathrm{Re}(\beta)] \in \Sigma^{m-1}(\pi_1(X))$. Assume that X is aspherical. Let $\hat{X}$ be the universal cover of X and let $f : \hat{X} \to \mathbb{R}$ be a primitive of the lift of β to $\hat{X}$. We set $\hat{X}_d = f^{-1}([d, \infty))$. We shall prove that

$$\pi_i(\hat{X}_d) = 0$$

for $i \leq m - 2$ and every real number d; this obviously implies that $\hat{X}_d$ is essentially $(m-2)$-connected. theorem 48 implies that $\hat{X}_d$ is path-connected for all d. So let $1 \leq i \leq m - 2$ and $\xi : S^i \to \hat{X}_d$ be a continuous map representing a class in $\pi_i(\hat{X}_d)$. Since $\hat{X}$ is contractible, ξ extends to a continuous map $\bar{\xi} : B^{i+1} \to \hat{X}$. If $c := \min\{d, \inf_{B^{i+1}} f \circ \bar{\xi}\}$, the class of ξ vanishes in $\pi_i(\hat{X}_c)$. Since by theorem 48 the inclusion $\hat{X}_d \subset \hat{X}_c$ induces an isomorphism on π_i, we see that $[\xi] = 0$ in $\pi_i(\hat{X}_d)$. Hence $\pi_i(\hat{X}_d) = 0$. This proves that the class $b = [\mathrm{Re}(\beta)]$ belongs to $\Sigma^{m-1}(\pi_1(X))$ (hence to $-\Sigma^m(\pi_1(X))$ as well).

Here is a direct argument that the kernel of b is of type $\mathcal{F}_{m-1}$ if b is rational. If b is rational, the image of the integration morphism $\pi_1(X) \to \mathbb{R}$ associated to b is cyclic. Consider the associated infinite cyclic covering space $X_0 \to X$ and let $g : X_0 \to \mathbb{R}$ be a primitive of the lift to X_0 of the form $\mathrm{Re}(\beta)$. Rationality of b implies that the critical values of g are discrete. The map g being proper, each critical level set contains only finitely many critical points. Let c be a regular value of g. We can choose an ascending sequence of compact intervals

$$I_0 = \{c\} \subset I_1 \subset \cdots \subset I_j \subset \cdots$$

such that $\bigcup_{j \geq 0} I_j = \mathbb{R}$ and $I_j \setminus I_{j-1}$ contains a single critical value of g. Since the critical points of g are isolated, Lefschetz theory implies that $g^{-1}(I_j)$ has the homotopy type of $g^{-1}(I_{j-1})$ with finitely many m-cells attached to it. Thus,

X_0 has the homotopy type of a space obtained from $g^{-1}(c)$ by attaching (possibly infinitely many) m-cells. Since $g^{-1}(c)$ is a compact manifold and X_0 is a $K(\ker(b), 1)$, we deduce that $\ker(b)$ is $\mathcal{F}_{m-1}$. It is not $\mathcal{F}_m$ by proposition 45. $\square$

We could also finish by appealing to Lueck's theorem on mapping tori and the fact that locally symmetric manifolds satisfy the Singer conjecture, but I thought that this approach is a bit more geometric and more importantly was too lazy to write about L^2-betti numbers.

We need one final technical lemma for theorem 12. Recall that a G_δ is by definition a countable intersection of open sets.

Proposition 49. *Let A be a complex torus. Let $U \subset H^0(A, \Omega^1_A)$ be the set of holomorphic 1-forms which do not vanish on any nontrivial subtorus. Then U contains a dense symmetric G_δ. Consequently, the set $O = \{a \in H^1(A, \mathbb{R}) \mid a = [\mathrm{Re}(\alpha)]$ with $\alpha \in U\}$ contains a dense symmetric G_δ of $H^1(A, \mathbb{R})$.*

Proof. We identify A with $\mathbb{C}^n/\Lambda$ where $n = \dim_C A$ and $\Lambda < \mathbb{C}^n$ is a lattice. The space $H^0(A, \Omega^1_A)$ is then identified with the dual space $(\mathbb{C}^n)^*$. We define:

$$U_0 = \{\phi \in (\mathbb{C}^n)^* \mid \phi(\gamma) \neq 0, \forall \gamma \in \Lambda - \{0\}\}.$$

The set U_0 is obviously a G_δ and we shall check that $U_0 \subset U$. Let $\phi \in (\mathbb{C}^n)^*$ be a holomorphic 1-form on A which vanishes on a subtorus $T \subset A \cong \mathbb{C}^n/\Lambda$ of positive dimension. Then the inverse image of T in $\mathbb{C}^n$ is a linear subspace V such that $\phi(V) = 0$ and $V \cap \Lambda < V$ is a lattice. This implies that ϕ vanishes on a nontrivial element of Λ, hence $\phi \notin U_0$. This concludes the proof. $\square$

Proof of theorem 12. Let $\Gamma < \mathrm{PU}(m, 1)$ be a torsion-free cocompact arithmetic lattice of the first type. Kazhdan proved that an arithmetic lattice of first type has a congruence subgroup with positive first Betti number. Applying Kazhdan's result and then theorem 40 we obtain a finite index subgroup $\Gamma_0 < \Gamma$ such that the Albanese map of the manifold

$$X_{\Gamma_0} = B/\Gamma_0$$

is an immersion. Fix a finite index subgroup $\Gamma_1 < \Gamma_0$. The Albanese map of X_{Γ_1} is also an immersion since there is a natural commutative square

$$\begin{array}{ccc} X_{\Gamma_1} & \longrightarrow & \mathrm{Alb}(X_{\Gamma_1}) \\ & & \downarrow \\ X_{\Gamma_0} & \longrightarrow & \mathrm{Alb}(X_{\Gamma_0}). \end{array}$$

theorem 46 implies that if α is a holomorphic 1-form on $\mathrm{Alb}(X_{\Gamma_1})$ which does not vanish on any nontrivial subtorus, the class

$$a = [\mathrm{Re}(\mathrm{alb}^*_{X_{\Gamma_1}} \alpha)]$$

lies in $\Sigma^{m-1}(\pi_1)$. Since the set of such classes is dense by proposition 49, this concludes the proof of the first statement of the theorem. The statement about rational classes follows from proposition 45. $\square$

The authors believe this is true in the non-cocompact case. In fact, the proof would work for any arithmetic lattice with positive first betti number, but it is not known whether second type or mixed type lattices generally have positive virtual first betti number.