Groups acting on trees

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1 Introduction

This expository essay will discuss the basics of Bass-Serre theory of groups acting on trees, with a view to proving the 'fundamental theorem' of this field: Suppose a group G acts on a tree X without inversion. Then the quotient space $Y = G/X$ is a graph of groups with fundamental group G. As the fundamental group of a graph of groups is an amalgamated free product of a series of smaller groups, this gives a lot of insight into the algebraic structure of the original group G, including of course how it can be 'built up' from smaller groups with amalgamation.
All of the above terminology will be defined in due course, and this essay will in general aim to be as self-contained as possible, though proofs will inevitably be omitted for some technical results or because of length constraints. For ease of understanding, sometimes remarks which aim to build intuition will appear in place of an actual proof. The assumed knowledge will be anything that appears in the undergraduate schedules for the Cambridge Mathematical Tripos. The main reference is the text [1] (or rather a translation of it), written by a pioneer of the theory and great giant of mathematics Jean-Pierre Serre.

2 Amalgamation

There are many ways of creating new groups from old ones. The most obvious one is the direct product, where the combined groups don't interact with each other. Then there is the semi-direct product, where one of the groups acts on the others, of which the direct product is a special case, and the wreath product which is useful in the study of the symmetric groups. However, any introductory course on algebraic topology should include the Seifert-Van Kampen theorem, which roughly says that if I glue two 'nice enough' spaces together, a loop in the product space can be written as some combination of loops in each individual component space. However, we would like loops in the intersection of the components, along which they are glued, to be counted the same, since in the new space that's, well, the same. That's what gluing means. This then motivated the idea of amalgamation, which turns out to be quite a useful notion. This idea will be pushed a little further in this section.
Formally, the result of amalgamation, called the amalgam, is defined as a direct limit. A direct limit of groups $\{G_i\}$ and a set homomorphisms F_{ij} from $G_i$to G_j is a group $G = \varinjlim G_i$, with homomorphisms $f_i : G_i \mapsto G$ such that $f_i \circ f = f_j$ for all $f \in F_{ij}$. Additionally, the following universal property is required: for any other group H, given homomorphisms $h_i : G_i \mapsto H$ with $h_j \circ f = h_i$ for all $f \in F_{ij}$, there is exactly one homomorphism $h : G \mapsto H$ such that $h_i = h \circ f_i$.

The uniqueness follows from playing with the universal property, and the existence can be made by taking presentations for the G_i, taking the union of generators and quotient out by the relations within the G_i and also quotienting out $y^{-1}f(x)$ for $y \in G_j, x \in G_i$ such that there is some f with $y = f(x)$.
Note that more generally, the 'glued together bits' need not be isomorphic: Given groups A, G_1, G_2 and homomorphisms $f_i : A \to G_i$, the direct limit is called the amalgam $G_1 *_A G_2$. On the other hand, given a group A, a family of groups $(G_i)_{i\in I}$ and an injection $f_i : A \to G_i$. Taking a direct limit of this family gives a sum of the G_i with A *amalgamated.*
Examples:

1. The infinite dihedral group $D_\infty \equiv \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$. With $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ generated by x, y subject to $x^2 = 1 = y^2$, $a = y, b = yx$ satisfy the dihedral group relations $a^2 = 1, aba^{-1} = b^{-1}$.
2. For coprime integers $n, m > 1$ and the natural homomorphisms from $\mathbb{Z}, \mathbb{Z}/n\mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}$ is the trivial group.(Consider the image t in the amalgam of $1 \in \mathbb{Z}$)
3. Taking A to be the trivial group in the last line before the examples, the sum is denoted $*G_i$ and called the free product. The free group F_n is n distinct copies of $\mathbb{Z}$ amalgamated this way.

A good question would be which groups are amalgams (in a non-trivial way). We will not have time to answer that in this essay, but that question is explored in the reference text [1]. It is intimately connected with the possible actions of the group on trees; the example given is that $SL_3(\mathbb{Z})$ is not an amalgam.
There is a structure theorem for amalgams which we will not have space to prove, but for completeness sake they are stated here. From here on assume A injects into G_i and consider the sum $*_A G_i$ with A amalgamated.
For all $i \in I$ choose a set S_i of right coset representatives of A in G_i, with $1 \in S_i$. Let $\boldsymbol{i} = (i_1, i_2, \cdots i_n)$ be a sequence of elements of I such that $i_m \neq i_{m+1}$ for all $1 \leq m \leq n-1$.
A reduced word of type $\boldsymbol{i}$ is any sequence $m = (a; s_1, \cdots s_n)$ where $a \in A, s_j \in S_j \backslash \{1\}$. Denote by f, f_i the canonical homomorphisms of A, G_i into $G = *_A G_i$. Then the result is

Theorem 2.1. *For all $g \in G$ there is a unique sequence $\boldsymbol{i}$ and reduced word m of that type such that*
$$g = f(a) f_{i_1}(s_1) \cdots f_{i_n}(s_n)$$

2.1 HNN

An important result due to Higman, B.H. Neumann, and H. Neumann is that for any group G and two distinct copies of a subgroup $A \leq G$, there is a group G' containing a subgroup isomorphic to G such that the two copies of the group A are conjugate in G'. Formally,

Proposition 2.2. *Let A be a subgroup of a group G and $\theta : A \to G$ an injective homomorphism. Then there is a group G' containing G and an element $s \in G'$ such that $\theta(a) = sas^{-1}$ for all $a \in A$. Furthermore, if G is countable, finitely generated, or torsion-free, then G' can be made to have the same property.*

Proof. This proof, or something along these lines, is probably what the average man on the street would come up with if you stopped them and demanded a proof. Take the infinite cyclic group S generated by an element s, and take the free product $G_1 = G * S$. In true caveman fashion, just *make* things conjugate by quotienting out the right things. Let N be the normal subgroup generated by the relations $\{sas^{-1}\theta(a)^{-1}\ a \in A\}$. Then G_1/N is a group with the desired property. This also makes it clear why G' has the corresponding property, e.g. for countability note that the free product contains only words of *finite* length. ☐

As an example, taking $G = A$ to both be the trivial group, then H is trivial and the semi-direct product with $\mathbb{Z}$ is just $\mathbb{Z}$, so $\mathbb{Z}$ is the HNN extension of the trivial group.
More proofs are always welcome in mathematics, and since HNN groups will reappear later as amalgams it is worth giving a proof involving amalgams.

Proof. For each $n \in \mathbb{Z}$ let $A_n = A, G_n = G$. Let H be the group obtained from amalgamating the G_n by means of the injections $\theta : A_n \to G_n$ and the canonical inclusion $A_n \to G_n$. Let u_n be the canonical homomorphism $G_n \to G_{n+1}$.

$$\begin{tikzcd}
\cdots G_{n-1} & & G_n & & G_{n+1} & & G_{n+2} \cdots \\
& A_n \arrow[ul, "1"] \arrow[ur, "\theta"'] & & A_{n+1} \arrow[ul, "1"] \arrow[ur, "\theta"'] & & A_{n+2} \arrow[ul, "1"] \arrow[ur, "\theta"'] &
\end{tikzcd}$$

Putting these together induces an automorphism $u : H \to H$. Now consider G_0. $\theta(a) \in G_0$ is precisely $u(a) \in G_1$ in the amalgam H, so the restriction of u to A is θ. Consider the semi-direct product $G' = H \rtimes S$, where S is the infinite cyclic group generated by an element s that acts on H by the automorphism u. Then G' is the desired group and s the required element. ☐

While the groups constructed in the two proofs look very different, exploiting the universal property of the free product with the natural inclusions of G, S into G' in the second proof gives a unique homomorphism from the free product into G'. One can then check that the kernel is normal subgroup given in the first proof. G' is sometimes called the group derived from (A, G, θ) by the HNN construction. There is quite a surprising consequence of this construction:

Corollary 2.3. *Every group G can be embedded into a group K such that any two elements in K of the same order are conjugate in K.*

Proof. Given a group G, repeatedly use the HNN construction to make successive pairs of elements conjugate to each other in some larger group. Passing to direct limits, this gives a group $E(G)$ such that any two elements of G of the same order are conjugate in $E(G)$. Repeating this to get $E(E(G)), E(E(E(G)))$ and so on and taking a direct limit of these groups, the direct limit is the desired group K. ☐

In particular, starting with a torsion-free group like $\mathbb{Z}$ gives a group K where any two non-identity elements are conjugate, a 'very non-abelian' group in some sense. This also gives an overkill solution to 'find an infinite simple group', since if K is a non-trivial normal subgroup then every element is a conjugate of every

non-identity element of K.
More importantly, this illustrates the power of 'just do it', in the same vein just quotienting out until we get what is needed as the first proof of the HNN construction, of just going along as far as you can in trying to construct a proof or an object with some property. Imagine if you had tried to write down the group above directly...

3 Graphs and Trees

3.1 Graphs

Combinatorial graphs are objects that many are familiar with, perhaps even from their maths olympiad days. Hence, we will not dwell too much on the formal definition. Briefly, a graph consists of a set of vertices V, and a set of edges E. For our purposes, graphs will be oriented graphs, with every edge having a corresponding reversed edge, i.e. a second edge with the arrow pointing in the opposite direction. All graphs considered will have a map $e \mapsto \bar{e}$, sending each edge to its reverse, with $\bar{\bar{e}} = e$. The set $\{e, \bar{e}\}$ will be called a *geometric edge*. Each edge e will also have associated to it the originating vertex $o(e)$ and the terminating vertex $t(e)$ in the natural way. Note that these need not be distinct, unlike in certain combinatorial settings. These newly defined graphs can be visualized by drawing graphs as one normally would with the understanding that each edge actually corresponds to a pair of directed edges.
The notions of path, cycle, tree and so on are defined as usual, as long as the visualization is a path, cycle or tree in the usual sense. The length of a path c will be denoted $l(c)$ A graph with no cycles of length ≤ 2 is known as a combinatorial graph. A cycle of length 1 is called a loop. By choosing exactly one of $e, \bar{e}$ for all edges, one can give the graph an *orientation* E_+. The edges not chosen form an opposite orientation known as E_-. A morphism from a graph (V, E) to a graph $(V'; E')$ is a mapping $\alpha : V \to V'$ which takes edges to edges and the origin and terminus of $\alpha(e)$ are, respectively, the images of the origin and the terminus of e. Graphs are isomorphic with the usual definition.

Definition. For any group G and subset S of G, the Cayley graph $\Gamma(G, S)$ is the graph with vertices labelled as elements of G, and an orientation $E_+ = G \times S$. Edges $e_{g,s}$ are defined by $o(g, s) = g, t(g, s) = gs$ for all $g \in G, s \in S$.

The Cayley graph is a fundamental object in algebraic topology and geometric group theory, so it may already be familiar to some readers. The first proposition we have is

Proposition 3.1. *Let $\Gamma = \Gamma(G, S)$ be a Cayley graph. Then:*

i The subgroup $< S >$ gives the connected component of Γ at the identity, and the left cosets of $< S >$ are in bijection with the connected components of Γ. From this it is immediate that $\Gamma(G, S)$ is connected if and only if $G =< S >$.

ii Γ contains a loop if and only if $1 \in S$.

iii Γ is a combinatorial graph if and only if $S \cap S^{-1} = \emptyset$.

Proof. (ii),(iii) follow easily from the definition of the edge set of the Cayley graph. For (i), note that a path from vertices g_1 to g_2 exists if and only if there exists a sequence $\{s_i\}_{i=1}^n \in S \cup S^{-1}$ such that $g_1 = g_2 s_1 s_2 \cdots s_n$. Hence g_1, g_2 are in the same component of Γ if and only if they are in the same left coset of $< S >$. $\square$

Given a graph Γ and a set of vertices V_0, denote by $\Gamma - V_0$ the subgraph with vertex set $V \backslash V_0$ and edge set all of the edges of Γ with the ones where either origin or terminus belongs to V_0.

3.2 Trees

Call a path without backtracking a *geodesic*. Recall the following facts about trees:

1. A tree on n vertices has exactly $n-1$ geometric edges, and at least two leaves.

2. Between any two vertices P, Q there is a unique geodesic between P and Q.

Defining $d(P, Q)$ as the number of edges along this geodesic makes the tree a metric space. Picking a vertex P, for any other vertex Q there is a unique, finite path homeomorphic to $[0, n]$ between them, so the obvious 'straight line' homotopy $x \mapsto tx$ shows that the path between P and Q is contractible. Note that the homotopy from a point Q' on the path is compatible with that from Q, which combined with the fact that a tree is a union of its finite subtrees, or in this case, the paths from P to Q for Q a vertex of the tree, shows that any tree is contractible.

3.3 Subtrees of graphs

Proposition 3.2. *Let P be a leaf of a non-empty graph Γ. Then*

i Γ is connected if and only if $\Gamma - P$ is connected.

ii Every cycle of Γ is contained in $\Gamma - P$

iii Γ is a tree if and only if $\Gamma - P$ is a tree

iv If Γ is connected, a maximal tree Λ contains all its vertices.

The first three are obvious. The last is basically if Λ doesn't contain some vertex Q, then the connectedness gives a path from Λ to Q and hence a bigger tree. The existence of a maximal tree is a Zorny issue: order the subtrees by inclusion, then given a chain, an upper bound is the union of all the elements of the chain (since the union of nested trees is a tree), so the hypothesis of Zorn's lemma is satisfied. By staring hard enough at (iv), readers should be able to convince themselves that the following is true:

Proposition 3.3. *Let Γ be a connected, non-empty graph. Then it has the homotopy type of a bouquet of circles and is a tree if and only if it is contractible. In other words, Γ is a tree if and only if Γ/Λ is a tree.*

We will not give a formal proof, but thinking about a graph, after taking a maximal tree and treating it as/contracting it to a point, the rest of the edges are just loops. The graph is a tree if and only if there are no edges outside of the maximal tree.

4 Groups acting on trees

Having the title groups acting on trees, started by discussing groups and then moving on to discuss trees, we will now discuss groups acting on trees, to no one's surprise at all. As mentioned in the introduction, the reason this is interesting is that the geometric properties of the action give insight into the algebraic properties of the group.

Definition. A group G acts on a graph Γ if it sends edges to edges and vertices to vertices in the sensible way, i.e. the images of endpoints are the endpoints of images. Say that a group acts without inversion if, for any $g \in G, e \in E$, $ge \neq \bar{e}$.

Only groups acting without inversion will be considered here for the following reason: a quotient graph of Γ can be defined by taking the vertices to be the set of G-orbits of vertices of Γ and the edges to be the G-orbits of edges. This quotient graph is a graph only if there are no inversions, so that the edges $e, \bar{e}$ are actually distinct. More is true:

Lemma 4.1. *Let G act without inversions on a connected graph X. Every subtree T' of $G \backslash X$ lifts to a subtree of X.*

Proof. Let T_0 be a maximal subtree in the set of all subtrees of X which project injectively into T' (which exists by reasoning analogous to the previous Zorn's lemma argument). Let T'_0 be the image of T_0 in T'. By definition, $T'_0 \subset T'$. Suppose that $T'_0 \neq T'$. Then there is some edge $y' \in T' \backslash T'_0$. Connectedness of T' implies that there is a path back to T_0, so WLOG assume $o(y')$ is a vertex of T'_0. If $t(y') \in T'_0$, the path from $o(y')$ to $t(y')$ in T'_0 combined with y' gives a cycle in T', so $t(y') \notin T'_0$. Let y be a lift of y'. Since gy is also a lift of y', WLOG assume that $o(y)$ is in T_0. Then adjoining $P = t(y)$ gives a tree T_1 containing T_0 as a subtree, and $T_1 \to T'$ is also injective, contradicting maximality, so $T' = T'_0$. $\square$

A tree of representatives of $X \mod G$ is any subtree T of X which is a lift of a maximal tree from the quotient graph. Note that this need not be unique.

Definition. If Γ has a subgraph $\Delta \simeq G \backslash \Lambda$, the quotient graph from the action of G, then Δ is called a *fundamental domain* for $\Gamma \mod G$

Fundamental domains need not exist: consider $\mathbb{Z}/3$ acting on the cycle of length 9 C_9. The quotient graph is a cycle of length 3, but this isn't a subgraph of C_9. However, if it acts on a snowflake with 9 branches, then a fundamental domain does exist: pick 3 consecutive branches. This result generalises to:

Proposition 4.2. *Let G be a group acting on a tree T. A fundamental domain for $T \mod G$ exists if and only if $G \backslash T$ is a tree.*

Proof. If $G\backslash T$ is a tree the previous lemma gives a lift to T, so there is a fundamental domain. Conversely, if a fundamental domain exists, T has no cycles, so $G\backslash T$ is isomorphic to a subgraph of a tree, which is also a tree, so $G\backslash T$ must be a tree. $\square$

Define a segment to be two vertices with a single geometric edge between them, such as $\overset{P}{\circ}\!\xrightarrow{\;y\;}\!\overset{Q}{\circ}$. This is the simplest fundamental domain that can be meaningfully considered. It turns out that properties of the graph correspond to algebraic properties of the group acting on the graph in a sense that will be made precise by the next two lemmas. Unless otherwise specified, the context will be the following: a group G acts on a graph X with fundamental domain a segment T, labelled as in the previous diagram. $G_P, G_Q, G_y = G_{\bar{y}}$ are the stabilisers of the vertices and edges of T.

Lemma 4.3. *X is connected if and only if G is generated by $G_P \cup G_Q$.*

Proof. Let X' be the connected component of X containing the segment T. Let $G' = \{g \in G : gX' = X'\}$ be the stabiliser of X', and let G'' be the subgroup of G generated by $G_P \cup G_Q$. If $h \in G_P \cup G_Q$, it fixes at least one vertex so hT is in the same connected component as T, i.e. $hT \subset X'$. However, the connected component of hT is hX', so $hX' = X'$, i.e. $h \in G'$, so $G_P \cup G_Q \subset G'$, so $G'' \subset G'$. The key insight is now that X is the disjoint union of $G''T$ and $(G\backslash G'')T$. If not, then there would be $x \in G''$, $y \in G\backslash G''$ with $xT = yT$, or equivalently $y^{-1}xT = T$. Either $y^{-1}xP = P$, in which case $y^{-1}x$, and hence y, would be in G'', or $y^{-1}xP = Q$ and $y^{-1}xQ = P$, which is forbidden since this would mean G acts with inversion on the graph. In either case, a contradiction is reached. Since $X' \subset G''T$, this implies $G' \subset G''$, so $G'' = G'$. The graph is connected if and only if $X = X'$, i.e. $G = G' = G''$ (otherwise $g \in G\backslash G'$ would give gX' a separate connected component). $\square$

Lemma 4.4. *X contains a cycle if and only if $G_P *_{G_y} G_Q \to G$ is not injective.*

Proof. A cycle is a sequence of edge $w_0, w_1, \cdots w_n$ s without backtracking such that $o(w_0) = t(w_n)$. Each w_i can be written in the form $h_i y_i$, where $y_i \in y, \bar{y}, h_i \in G$. Projecting down to the quotient T, $\bar{y_i} = y_{i-1}$ for $1 \leq i \leq n$. Let $P_i = o(y_i) = t(y_{i-1})$. Now,

$$h_i P_i = h_i o(y_i) = o(h_i y_i) = t(h_{i-1} y_{i-1}) = h_{i-1} t(y_{i-1}) = h_{i-1} P_i$$

so $h_i = h_{i-1} g_i$ for some $g_i \in G_{P_i}$. This makes sense geometrically since moving along the cycle edge by edge, one end of the edge is fixed every time, so it is a 'shift' by an element of the corresponding stabiliser. Furthermore, $h_i \notin G_y$ since otherwise,

$$\bar{w_i} = \overline{h_i y_i} = \overline{h_{i-1} g_i y_i} = \overline{h_{i-1} y_i} = h_{i-1} y_{i-1} = w_{i-1}$$

contradicting the no backtracking condition. Being a cycle is equivalent to $t(y_n) = P_0$, or alternatively taking indices modulo n that $h_0 P_0 = h_n P_0 = h_0 g_1 \cdots g_n P_0$, i.e. $g_1 g_2 \cdots g_n \in G_{P_0}$. Then there exists $g_0 \in G_{P_0}$ such that $g_0 \cdots g_n = 1$. Then X contains a cycle if and only if such a sequence of P_i whose projections alternate between P and Q and a sequence of $g_i \in G_{P_i} \backslash G_y$ exists. There are no cancellations precisely between the g_i because no elements are in the intersection of the vertex stabilisers, i.e. the edge stabilisers, so the statement $g_0 \cdots g_n = 1$ is equivalent to the homomorphism not being injective. □

These two lemmas can then be put together to deduce

Theorem 4.5. *X is a tree if and only if the homomorphism $G_P *_{G_y} G_Q \to G$ induced by the natural inclusions of the vertex stabilisers into G is an isomorphism.*

since the defining property of a tree is that it is connected and has no cycles. Being connected is equivalent to the homomorphism being surjective, and having no cycles is equivalent to the homomorphism being injective. Conversely, every amalgam acts in this way:

Theorem 4.6. *Let $G = G_1 *_A G_2$ be an amalgam of two groups. Then there is a unique tree up to isomorphism on which G acts with fundamental domain a segment T, so that the stabilisers are $G_P = G_1, G_Q = G_2, G_y = A$.*

Proof. If you sit down and try and come up with such a graph, it becomes quite clear what the graph is and why it is unique up to isomorphism. Starting with a segment T, letting G act on it gives that the orbit of P is in bijection with the left cosets of G_1. Similar reasoning for Q and y then gives that the graph must be isomorphic to the graph Γ with:

$$V(\Gamma) = G/G_1 \bigsqcup G/G_2, E(\Gamma) = G/A \bigsqcup \overline{G/A}$$

with $o(gA) = gG_1, t(gA) = gG_2$. The obvious action of G gives the statement about stabilisers. That Γ is a tree follows from the previous theorem. □

This establishes an equivalence between a group G being amalgam of two groups and G acting on a tree with a segment as fundamental domain. The more general case, with amalgams of more than two groups, is similar and is the main structure theorem of this essay. However, it is analogous, and readers should note the similarities in theorem statements and proof ideas.
Having said that, we now prove a generalisation of X connected implies G generated by vertex stabilisers that will be needed in the more general case. We're on the topic anyway so why not?

Lemma 4.7. *Let G be a group acting on a connected graph X, and let T be a tree of representatives of $X \mod G$. Let Y be a subgraph of X containing T such that every edge of Y has an extremity in T and $G \cdot Y = X$. For each edge e of Y with origin in T let g_e be an element of G such that $g_e t(e) \in V(T)$. Then the group H generated by the vertex stabilisers $G_P, P \in V(T)$ and the g_e is G.*

Note that the assumption of being a tree of representatives only guarantees $G \cdot Y$ contains all the vertices, but not necessarily all the edges, so $G \cdot Y = X$ isn't redundant. It should also be obvious that this is a generalisation since the g_e are all elements of the vertex stabilisers in the case of a segment.

Proof. It suffices to show that $H \cdot V(T) = V(X)$. Since H contains the g_e, the hypothesis that all the terminal vertices can be translated to T implies that $V(Y) \subset H \cdot V(T)$. It now suffices to show that $H \cdot Y = X$, and since X

is connected this reduces to showing that any edge e with origin in $H \cdot Y$ is contained in $H \cdot Y$. By translating with an element of H if necessary, assume $P = o(e) \in V(T)$. Now the assumption $G \cdot Y = X$ implies there is $g \in G$ such that $ge \in Y$ and it suffices to show that $g \in H$.
Since $ge \in Y$, either $o(ge)$ or $t(ge)$ is a vertex of T. In the first case, P and gP are two vertices of T congruent $\mod G$ so they are equal and $g \in G_P$. In the second case, $y = \overline{ge}$ has its origin in T, so there is some g_y such that $g_y t(y) \in V(T)$ and as in the first case, P must coincide with $g_y t(y) = g_y o(ge) = g_y g P$ so $g \in g_y^{-1} G_P \subset H$. □

4.1 Groups acting freely on trees

In this slight detour more examples of how the geometric properties of a group action can be used to deduce algebraic facts about the group are given. It should provide some more practice for when this is done again in the future, but more importantly is interesting maths, which is good (in my book at least).

Proposition 4.8. *Let $X = \Gamma(G, S)$ be the Cayley graph defined by a group G and a subset S. Then X is a tree if and only if G is a free group with basis S*

Proof. If G is a free group with basis S, then every element can be written as a reduced word $g = s_1^{e_1} \cdots s_n^{e_n}$ with $e_i = \pm 1$. Let G_n denoted the set of words of length at most n, then this gives a map $G_n \to G_{n-1}$ by just chopping off the last symbol. It is easy to see that X is built up successively by the G_n, and by the result on Cayley graphs there is no loop since $1 \notin S$.
Conversely suppose X is a tree. By the same result on Cayley graphs S generates G and $S \cap S^{-1} = \emptyset$. Suppose for a contradiction that a reduced word of positive length in $F(S)$ (the free group on was S as basis) is the identity in G. Pick such a reduced word $g = s_1^{e_1} \cdots s_n^{e_n}$ of minimal length. Let P_i be the vertex associated to $s_1^{e_1} \cdots s_i^{e_i}$. The hypotheses imply $n \geq 3$, and minimality implies that any consecutive vertices P_i, P_{i+1} are distinct. Then the P_i form a cycle, contradicting the assumption that X is a tree. □

Say that a group acts *freely* on a graph if, in addition to acting without inversions, each vertex is fixed only by the identity. For example, in the proposition above G acts freely on X, which shows that every free group acts freely on some tree. A converse is easy to see: if G acts freely on a tree then it is a free group. It suffices to check that any non-trivial reduced word is not the identity, and this is true since if not, one would generate a cycle and get a contradiction as before. There are some details to check, which we won't since there is a stronger result that we prove instead.

Theorem 4.9. *Let a group G act freely on a tree X, T be a tree of representatives of $X \mod G$ and E_+ be an orientation preserved by G.*

i Let S be the set of non-identity elements in G for which there is an edge $e \in E_+$ with origin in T and terminus in gT. Then S is a basis for G.

ii If $X^* = G \backslash X$ has a finite number of vertices v and X^* has a geometric edges, then $|S| - 1 = a - s$. (Note this means $|E(X^*)| = 2a$.)

Proof. (i): G acts freely, and T injects into X^* naturally, so $g \mapsto gT$ is a bijection of G onto the set of (pairwise disjoint) translates of T. Let X' be the graph formed from X by shrinking each gT to a single vertex denoted (gT). As proved in a previous proposition shrinking trees to vertices leaves a graph which is also a tree, so X' is a tree.
Then $(gT) \mapsto g$ can be thought of as a bijection $\alpha : V(X') \to V(\Gamma(G, S))$ since the vertices of the Cayley graph $V(\Gamma(G, S))$ are in bijection with elements of G. At this point a fairly natural thing to try is to extend α to a graph isomorphism. Give X' the orientation E'_+, the orientation induced from E_+ (well-defined since the edges of X' are a subset of the edges of X). Now define $\alpha : E'_+ \to G \times S$ to give an orientation on the Cayley graph. Let e be an edge of X'. Since the trees have all been contracted to points, the origin and terminus of e are some vertices $(gT), (g'T)$ respectively. Hence $s = g^{-1}g' \in S$, so set $\alpha(e) = (g, s)$. By definition of S, this is surjective. Injectivity follows from X' is a tree, so there won't be two edges between the same two vertices $(gT), (g'T)$, and that α is injective on vertices.
This shows α is indeed a graph isomorphism, and hence X' is a tree implies, by the previous result, that G is free with basis S.
(ii): Let W be the set of edges where the origin but not terminus is a vertex of T. By (i) this is in bijection with the basis S. But after deleting the $|S|$ geometric edges of X^* which are images of edges of W, we are left with precisely the geometric edges coming from the injection $T \to X^*$. This is a maximal tree in X^* with s vertices, hence $a - |S| = s - 1$. □

This establishes that groups are free if and only if they always act freely on trees and there is some tree on which they act freely. Immediately this shows that a subgroup of a free group is free, since the action of the free group on its Cayley graph induces a free action of its subgroups on that same Cayley graph. In fact, one can obtain an analogue of the Riemann-Hurwitz formula from algebraic topology for free groups.
Suppose S_1, S_2 are free bases for the same group G. By abelianising, i.e. quotient out all the commutators, this reduces to a question about $\mathbb{Z}$-modules, where it is much more easily seen to be true that $|S_1| = |S_2|$. Then we can write $r_G = |S_1|$ without worrying about having committed a cardinal sin of mathematics. Moreover,

Corollary 4.10. *Let G be a free group and H a subgroup of finite index n. Then $r_H - 1 = n(r_G - 1)$.*

Proof. Let $G_1 = G$, $G_2 = H$, and let Γ be the Cayley graph of G for some free basis S, on which G acts freely. Let $\Gamma_i = G_i \backslash \Gamma$, $s_i = |V(\Gamma_i)|$, and $a_i = |E(\Gamma_i)|$. This gives $s_2 = ns_1, a_2 = na_1, s_1 = 1$. Then rearrange the formula in (ii) of the theorem applied to the G_i. □

In particular, if $r_G = 2$ gives the possibility of having free subgroups of all countable ranks > 1, and in fact it does. For example, in the free group $F_2 =< a, b >$ the family $\{a^n b a^{-n}\}_{n \in \mathbb{N}}$ is a basis for a free group on countably infinitely many generators. This also shows that the same is true for the free groups of rank $n > 1$, since they all contain F_2 as a subgroup. Groups can be weird.

5 Trees of groups

In this section a generalization of previous results will be given. In particular, amalgamations of more groups, not necessarily all sharing a common subgroup, will be considered.

Definition. A *graph of groups* $(\mathbb{G}, T)$ consists of a graph T and a set of groups $\mathbb{G}$ such that there is a group G_P for every vertex P of T, a group $G_e = G_{\bar{e}}$ for every edge e, and an injective homomorphism sending $a \in G_e$ to its image $a^e \in G_{t(e)}$.

If T is a tree, then this is known as a tree of groups. Let $G_T = \varinjlim(\mathbb{G}, T)$ denote the direct limit of the groups, also known as the amalgam of the vertex groups G_P along the edge groups G_e. For example, in the case of T a segment as before, then the direct limit is just the amalgam $G_P *_{G_y} G_Q$. This also allows us to build up the direct limit of a tree of groups by adding in edges one at a time: if a vertex v is a leaf of a tree T, taking T' to be $T \backslash v$, $G_{T'} = \varinjlim(\mathbb{G}', T')$ where $\mathbb{G}'$ is the restriction of $\mathbb{G}$ to T' and $e, \bar{e}$ to be the edge connecting v to T', we see that $G_T = G_{T'} *_{G_e} G_v$.
Since the homomorphisms $G_P \to G_T$ and $G_e \to G_T$ are injective, from hereon the groups will be identified with their images in G_T. By taking direct limits, it suffices to check this for finite trees, where this follows by induction (c.f. 4.5).
Just as every segment is the fundamental domain of an amalgam of two groups, every tree of groups is a fundamental domain of a larger graph with the groups involved in the amalgamation as stabilisers of the respective vertices:

Theorem 5.1. *Let $(\mathbb{G}, T)$ be a tree of groups. There is a graph X containing T and an action of G_T on X, unique up to isomorphism, such that T is a fundamental domain for $X \mod G_T$ and for all $P \in V(T)$ (resp. $e \in E(T)$), the stabiliser of P (resp. e) in G_T is G_P (resp. G_y). Moreover, X is a tree, which will be called the graph* associated *with* $(\mathbb{G}, T)$.

Proof. The construction of X follows that given in the previous theorem for a segment: take $V(X)$ to be the disjoint union of G_T/G_P over all vertices P of T, do the same for every edge of T, and the assertions follow immediately as in the previous theorem, apart from X being a tree. Note that everything commutes with taking direct limits, so T can be taken to be the direct limit of its finite subtrees, and we consider the corresponding G_T, X associated with these finite subtrees. Hence we are reduced to the finite case, where we argue by induction on $n = |V(T)|$.
$n = 1$ is trivial since $X = T$, so assume $n > 1$. Then removing a leaf v and the corresponding edges $e, \bar{e}$ to get a subtree T', we get that $G_T = G_{T'} *_{G_e} G_v$. Let $X' = G_{T'} \cdot T'$. This is a subgraph of X, which is easily checked to be associated to (G, T'). By the induction hypothesis X' is a tree. Note that $g_1 X' = g_2 X'$ if and only if they are in the same left coset of $G_{T'}$ and are disjoint otherwise. Let $\tilde{X}$ be the graph derived from X by shrinking each tree gX' to a point. G_T acts on $\tilde{X}$ with fundamental domain the segment $T/T' = \overset{(T')}{\circ} \xrightarrow{\;y\;} \overset{P}{\circ}$. Then since $G_T = G_{T'} *_{G_e} G_v$, $\tilde{X}$ is also a tree by 4.5. Hence X is a tree. $\square$

A converse is also true: let a group G act on a graph X with fundamental domain a tree T. Let (G, T) be the tree of groups where the group associated

to each edge or vertex is the stabiliser in G of that edge of vertex. The injective map from the edge group to the vertex group is just the natural inclusion map. Let $G_T = \varinjlim(\mathbb{G}, T)$. The inclusion maps $G_P \to G$ extend to a homomorphism $G_T \to G$ which is surjective if and only if X is connected by 4.3.
Let $\tilde{X}$ denote the tree associated to $(\mathbb{G}, T)$. Sending $T \subset \tilde{X}$ to $T \subset X$ by the identity map extends uniquely to a morphism $\tilde{X} \to X$ which commutes with the map $G_T \to G$.

Theorem 5.2. *With the above hypotheses and notations, TFAE:*

i X *is a tree*

ii $\tilde{X} \to X$ *is an isomorphism*

iii $G_T \to G$ *is an isomorphism*

Proof. (iii) $\implies$ (ii) follows from the previous theorem. (ii) $\implies$ (i) since $\tilde{X}$ is a tree
(ii) $\implies$ (iii): let P be a vertex of T and let $(G_T)_P$ (resp. G_P) be the corresponding stabiliser of P in G_T (resp in G). By construction of the map, in particular requiring the action to be equivariant with respect to the map, $G_T \to G$ induces an isomorphism from $(G_T)_P$ to G_P. On the other hand, if $\tilde{X} \to X$ is a bijection, the kernel H of $G_T \to G$ is contained in $(G_T)_P$, so H must be trivial and the map is injective. $G_T \to G$ is surjective, since $\tilde{X}$, being a tree, is connected, and therefore so is X, so it must be an isomorphism.
(i) $\implies$ (ii): $G_T \cdot T = \tilde{X}$ and $G \cdot T = X$ implies $\tilde{X} \to X$ is surjective (by equivariance). On the other hand, the homomorphism $G_T \to G$ induces isomorphisms between the stabilisers of the corresponding vertices (and edges) of $\tilde{X}$ and X. Hence $f : \tilde{X} \to X$ is locally injective, i.e. injective on a set of edges with a common origin. We will need a lemma to finish off here, and the reader should make a mental note that this will reappear in the section on the Structure Theorem.

Lemma 5.3. *If* $\tilde{X}$ *is connected,* X *is a tree, and* $f : \tilde{X} \to X$ *is locally injective, then* f *is injective.*

Proof. Since $\tilde{X}$ is connected, it suffices to show that for any injective path c, $f \circ c$ is also injective. X is a tree, so injectivity can only fail by backtracking. But this never happens since c is injective and f is locally injective. ☐

With this, f is injective and surjective, so is an isomorphism. ☐

5.1 The Fundamental Group

Having dealt with the case where the quotient graph is a tree, the more general case will require a generalisation of amalgams that is known as the fundamental group of a graph of groups.

The group $F(G, Y)$

For a graph of groups $(\mathbb{G}, Y)$ with Y connected and non-empty, the group $F(\mathbb{G}, Y)$ is the group generated by the groups G_P and elements e for every edge, subject to the relations

$$\bar{e} = e^{-1}, ea^e e^{-1} = a^{\bar{e}}$$

if e is an edge and $a \in G_e$.
Given a path in Y whose origin is P_0, its edges will be denoted $e_1, e_2 \cdots e_n$ with $P_i = o(e_{i+1}) = t(e_i)$.

Definition. A word of type c in $F(\mathbb{G}, Y)$ is a pair (c, μ) where $\mu = (r_0, \cdots, r_n)$ is a sequence of elements $r_i \in G_{P_i}$. The element $|c, \mu| = r_0 e_1 r_1 \cdots y_n r_n$ of $F(\mathbb{G}, Y)$ is said to be associated to the word (c, μ).

At this point the fundamental group can be defined in one of two ways. Either:

Definition. Let P_0 be a vertex of Y. Let $\pi_1(\mathbb{G}, Y, P_0)$ be the set of elements of $F(\mathbb{G}, Y)$ of the form $|c, \mu|$ where c is a path starting and ending at P_0. This is easily seen to be a subgroup of $F(\mathbb{G}, Y)$, called the fundamental group of $(\mathbb{G}, Y)$ at P_0.

If the trivial group is associated to each vertex, the corresponding fundamental group is the fundamental group in the usual sense of the term of the graph based at P_0. In general, the homomorphism from each G_P to the trivial group $\{1\}$ induces a surjective homomorphism $\pi_1(\mathbb{G}, Y, P_0) \to \pi_1(Y, P_0)$. Alternatively, the group can be defined as:

Definition. Let T be a maximal tree of Y. The fundamental group $\pi_1(\mathbb{G}, Y, T)$ of $(\mathbb{G}, Y)$ at T is defined to be the quotient of $F(\mathbb{G}, Y)$ by the normal subgroup generated by the elements e corresponding to edges of T. Letting g_e denote the image of an edge e in π_1, $\pi_1(\mathbb{G}, Y, T)$ is the group generated by the G_P and g_e subject to $\bar{e} = e^{-1}, g_e a^e g_e^{-1} = a^{\bar{e}}$if e is an edge and $a \in G_e$, and $g_e = 1$ if $e \in E(T)$.

For example, in the case of a segment $T = \overset{P}{\circ}\xrightarrow{\;y\;}\overset{Q}{\circ}$, both definitions give the amalgam $\pi_1 = G_P *_{G_y} G_Q$ as the fundamental group. It would be very strange to give two definitions of the same thing and then for them to give different results wouldn't it? However, some care is needed. Relative to a maximal tree, then it really is $G_P *_{G_y} G_Q$ in the obvious way. Relative to $P_0 = P$ however, one instead obtains sequences of the form $p_0(yq_0y^{-1})p_1(yq_1y^{-1})\ldots$ for $p_i \in G_i, q_i \in G_Q$, i.e. the elements of G_Q are always conjugated by y (or y^{-1} depending on definition). Note however, that one can't go from $\pi_1(\mathbb{G}, Y, P_0)$ to $\pi_1(\mathbb{G}, Y, T)$ by quotienting out the normal subgroup generated by the y since y alone isn't even an element of $\pi_1(\mathbb{G}, Y, P_0)$.

Proposition 5.4. *Let* $(\mathbb{G}, Y)$ *be a graph of groups,* $P_0 \in V(Y)$ *and* T *a maximal tree. The canonical projection* $p : F(\mathbb{G}, Y) \to \pi_1(\mathbb{G}, Y, T)$ *induces an isomorphism from* $\pi_1(\mathbb{G}, Y, P_0)$ *to* $\pi_1(\mathbb{G}, Y, T)$

Proof. For $P \in V(Y)$ let c_P be the geodesic in T joining P_0 to P with edges $e_1, e_2 \cdots e_n$ in that order. Let $\gamma_P = e_1 e_2 \cdots e_n \in F(\mathbb{G}, Y)$. Let $x' = \gamma_P x \gamma_P^{-1}$ for $x \in G_P$ and $e' = \gamma_{o(e)} e \gamma_{t(e)}^{-1}$ for $e \in E(Y)$. Note that if $e \in T$, the geodesic to either $o(e)$ or to $t(e)$ must pass through e or $\bar{e}$ otherwise there would be a cycle, so for $e \in T, e' = 1$. Note also that $x', e' \in \pi_1(\mathbb{G}, Y, P_0)$ since for the edges e', the added elements close up a cycle passing through P_0.
(A motivation for this is the isomorphism between the two groups in the case of a segment above, i.e. that conjugating things suitably worked before, so why not now?).
Define $f : \pi_1(\mathbb{G}, Y, T) \to \pi_1(\mathbb{G}, Y, P_0)$ by $f(x) = x', f(y) = y'$. We show this is a homomorphism. It is easy to check that $f(x_1 x_2) = f(x_1) f(x_2)$ for x_1, x_2 in the same vertex group and $f(e_1 e_2) = f(e_1) f(e_2)$ for consecutive edges e_1, e_2, so all that is left is to make sure that the images obey the right relations. For all $a \in G_e$

$$\begin{aligned} e'(a^e)'e'^{-1} &= \gamma_{o(e)} e \gamma_{t(e)}^{-1} \gamma_{t(e)} a^e \gamma_{t(e)}^{-1} \gamma_{t(e)} e^{-1} \gamma_{o(e)}^{-1} \\ &= \gamma_{o(e)} a^{\bar{e}} \gamma_{o(e)}^{-1} \\ &= (a^{\bar{e}})' \end{aligned}$$

Since(γ_P) is a path in T, $p(\gamma_P) = 1$ so p strips away the added elements to give $p \circ f$ is the identity on $\pi_1(\mathbb{G}, Y, T)$. Let c be a closed path with origin P_0, edges $e_1, e_2 \cdots e_n$ and vertices $P_i = o(e_{i+1}) = t(e_i)$, $\mu = (r_0, r_1 \cdots r_n)$. Let $r_0 y_1 r_1 y_2 \cdots y_n r_n$ be the word associated to (c, μ). Let $r_i' = \gamma_{P_i} r_i \gamma_{P_i}^{-1}$, $e_i' = \gamma_{P_i} e_i \gamma_{P_i+1}^{-1}$. Note that $P_{n+1} = P_0$, and $\gamma_{P_0} = 1$, so $r_0' y_1' r_1' y_2' \cdots y_n' r_n' = r_0 y_1 r_1 y_2 \cdots y_n r_n$. Hence $f \circ p$ is the identity on $\pi_1(\mathbb{G}, Y, P_0)$. $\square$

Remarks:

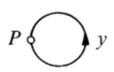
1. This shows that the fundamental group is well defined, and like the fundamental group of a topological space, is independent of either base point P_0 or the maximal tree T, which is a relief.

2. Let R be the normal subgroup of π_1 generated by the vertex groups G_P. Then π_1/R is the fundamental group in the sense of topological spaces of the graph Y relative to the maximal tree T (this can also be seen to be reducing all the vertex groups to the trivial group). It is a free group with basis g_e for $y \in E_+ \backslash T$ for some orientation E_+.

Examples:

1. If Y is a tree, $\pi_1(\mathbb{G}, Y, Y) = \varinjlim(\mathbb{G}, Y)$, generalising the case of a segment

2. Fundamental groups are in general amalgams, but although some groups aren't amalgams in non-trivial ways any group G can still be made a fundamental group of some graph of groups by taking a segment, and giving one vertex the group G and the other vertex the trivial group.

3. Take Y to be a loop 

. Let $A = G_y$. There are then two injections, along $y, \bar{y}$, from A to G_P. Since every path is a closed path, the fundamental group is just $F(\mathbb{G}, Y)$, generated by G_P and an element

$g = g_y$ such that $ga^y g^{-1} = a^{\bar{y}}$ for all $a \in A$. Now with θ as the map $a \mapsto a^{\bar{y}}$, this shows that $\pi_1(\mathbb{G}, Y, P)$ is the group derived from (A, G_P, θ) by the HNN construction. The proof then shows that π_1 is the semi-direct product of $< g >$ and the normal subgroup generated by the conjugates $G_n = g^n G_P g^{-n}$ of G_P as n ranges over the integers.

5.2 Reduced Words

Let (c, μ) be a word of type c, where c is a path with origin P_0 and edges $e_1, \cdots e_n$, $\mu = (r_0, r_1, \cdots r_n)$. Denote by G_e^e the image of G_e in $G_{t(e)}$

Definition. A word (c, μ) is said to be reduced if it satisfies the following condition: if $n = 0$, $r_0 \neq 1$; if $n \geq 1$, $r_i \notin G_{e_i}^{e_i}$ for each index i such that $y_{i+1} = \bar{y}_i$.

In particular, every word of type a path with length ≥ 1 without backtracking is reduced.

Theorem 5.5. *If (c, μ) is a reduced word, the associate element $|c, \mu|$ of $F(\mathbb{G}, Y)$ is $\neq 1$.*

Before the proof of this, the corollary below is given in preparation as it will be needed for the proof. Don't worry, maths isn't broken. It's just that the main proof relies on proving special cases and using the corollary for those cases to deduce the general case.

Corollary 5.6. *i The homomorphisms $G_P \to F(\mathbb{G}, Y)$ are injective*

ii If (c, μ) is reduced and $l(c) \geq 1$, then $|c, \mu| \notin G_{P_0}$.

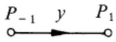
iii If T is a maximal subtree and (c, μ) has the type of a cycle, the image of $|c, \mu|$ in $\pi_1(\mathbb{G}, Y, T)$ is non-trivial.

Proof. (i) is just the statement of the theorem with $l(c) = 0$
For (ii), suppose not. Then define $\mu' = (|c, \mu|^{-1} r_0, r_1 \cdots r_n)$, so that $|c, \mu'| = 1$ contradicting the theorem.
For (iii), note $|c, \mu| \in \pi_1(\mathbb{G}, Y, P_0)$, so is non-trivial there, and there is an isomorphism from $F(\mathbb{G}, Y)$ to $\pi_1(\mathbb{G}, Y, T)$ which induces an isomorphism from $\pi_1(\mathbb{G}, Y, P_0)$ to $\pi_1(\mathbb{G}, Y, T)$. □

The proof is on the long side and rather technical so only a very brief sketch of the ideas is given below.

Proof. (very sketchy) For any graph Y, let Y' be a connected, non-empty subgraph of Y. This induces a graph Y/Y' by shrinking Y' to a single vertex. Giving this vertex the group $F(\mathbb{G}|_{Y'}, Y')$ gives a new graph of groups $(\mathbb{H}, W)$. $F(\mathbb{G}, Y)$ can be seen to be isomorphic to $F(\mathbb{H}, W)$ by realising that this amounts to saying you can build up loops and words bit by bit, first inside Y' then adding in the stuff from outside. Moreover, the map is induced by the shrinking in a fairly natural way.
Assuming that the theorem is true for $(\mathbb{H}, W)$, one can then show that if $|c, \mu|$ is reduced in $(\mathbb{G}, Y)$ then the $|c', \mu'|$ of $(\mathbb{H}, W)$ is also reduced. This then shows that the theorem is true for $(\mathbb{G}, Y)$. Applying this reduction then reduces us to

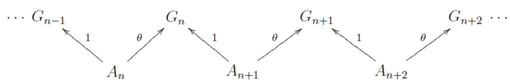
the case of a segment and a loop.

For a segment 

and an element of the form $r_0 y^{e_1} r_1 \cdots y^{e_n} r_n$ with $e_i = \pm 1, e_{i+1} = -e_i, r_i \in G_{P_{e_i}} \backslash G_y^{y_{e_i}}$. There is a canonical homomorphism from $F(\mathbb{G}, Y)$ to $\pi_1(\mathbb{G}, Y, Y) = G_{P_1} * G_{P_{-1}}$ sending the word to $r_0 r_1 \cdots r_n$. Amalgams are often useful for showing that groups are non-trivial, due to the structure theorem for amalgams that proves elements are non-trivial. Length constraints mean that result couldn't be proved, but it is used here to show that the image is non-trivial. This should feel believable without a formal proof.
By induction the theorem is true for finite trees, and by taking direct limits true for all trees.

Previously, we saw that the fundamental group of a loop 

is an HNN group, and if R is the normal subgroup generated by G_0. Setting $G_n = y^n G_0 y^{-n}$, we see that $y a^y y^{-1} = a^{\bar{y}}$ extends to $y^n a^y y^{-n} = y^{n-1} a^{\bar{y}} y^{1-n}$ so G_{n-1} has a subgroup isomorphic to A that is also a subgroup of G_n. (Recall the diagram below from when HNN groups were introduced). Amalgamating



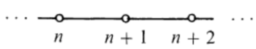
along these gives R, since the only way to 'conjugate out of G_0 is to use y to get to the conjugates G_n. Elements are once again of the form $r_0 y^{e_1} r_1 \cdots y^{e_n} r_n$ with $e_i = \pm 1, e_{i+1} = -e_i, r_i \notin A_{y^{e_i}}$ if $e_{i+1} = -e_i$.
If $\sum e_i \neq 0$ then the word $|c, \mu|$ isn't in R so is non-trivial, but really it's because if a path loops around a non-zero number of times it obviously can't be the identity. Assuming $\sum e_i = 0$, set

$$d_i = e_1 + \cdots e_i, s_i = y^{d_i} r_i y^{-d_i}$$

allows us to rewrite $|c, \mu|$ as $s_0 s_1 \cdots s_n$ with $s_i \in G_{d_i}$ with $d_0 = 0 = d_n$. Note that $d_{i+1} - d_i = e_{i+1}$, and the condition that $e_{i+1} = -e_i$ is equivalent to $d_{i+1} = d_{i-1}$, so the condition $r_i \notin A_{y^{e_i}}$ if $e_{i+1} = -e_i$ translates into

$$s_i \notin y^{d_i} A_{y^{e_i}} y^{-d_i}$$

Now consider the path/tree whose vertices bijection with $\mathbb{Z}$ and has edges between consecutive numbers. Let (K, T) be the tree of groups associating G_n to



the vertex n. R is the fundamental group, and applying the corollary to the word $s_0 s_1 \cdots s_n$ in R gives the result as this is the word associated to a *closed path* in T.
This now allows us to prove the theorem for all finite graphs, and taking direct limits gives the general result. $\square$

6 Structure of groups acting on trees

Many parallels have been drawn in the previous section between the fundamental group of a graph of groups and that of a topological space. Geometrically, if a group G acts on a simply-connected Hausdorff topological space X by homeomorphisms freely and properly discontinuously, then $\pi_1(G\backslash X, [x]) \simeq G$. This gives an equivalence between the fundamental group and an action of this group on the universal cover. This section will develop the analogous results for fundamental groups of graphs of groups.

6.1 Universal Coverings

Let $(\mathbb{G}, Y)$ be a graph of groups with Y connected and non-empty, T a maximal tree of Y and E_+ an orientation of Y. Let 1_e be the indicator function for an edge e of whether it is in the orientation, and let $|e|$ be the edge among $e, \bar{e}$ which is in E_+. By analogy with the case for general topological spaces, we seek a graph $\tilde{X} = \tilde{X}(\mathbb{G}, Y, T)$ such that

- There is an action of $\pi = \pi_1(\mathbb{G}, Y, T)$ on $\tilde{X}$
- There is a morphism $p : \tilde{X} \to Y$ inducing an isomorphism from $\pi\backslash\tilde{X}$ to Y.
- Y embeds into $\tilde{X}$ with $P \in V(Y) \mapsto \tilde{P}$, $e \in E(Y) \mapsto \tilde{e}$

For $P \in V(Y)$ we require the stabiliser π_P of $\tilde{P}$ in π to be G_P. Similarly, if $e \in E(Y)$ with $w = \overline{|e|}$, the stabiliser $\pi_{\tilde{e}}$ of w should be the subgroup G^w_w of $G_{t(w)}$. As before, this forces the vertices and edges to correspond to orbits, so set

$$V(\tilde{X}) = \bigsqcup \pi/\pi_P, E(\tilde{X}) = \bigsqcup \pi/\pi_{\tilde{e}}$$

Taking the cosets all corresponding to 1 gives the embedding. Recall that g_e denotes the image of e in π. Define

$$\begin{aligned} \overline{g\tilde{e}} &= g\tilde{\bar{e}} \\ o(g\tilde{e}) &= gg_e^{1_e - 1}\tilde{o(e)} \\ t(g\tilde{e}) &= gg_e^{1_e}\tilde{t(e)} \end{aligned}$$

The reason the slightly funny looking extra bits are included is so that the next result is true:

Lemma 6.1. *These expressions are well-defined, i.e. they depend only on the coset that g is in.*

Proof. The LHS depends only on which coset in π/G^w_w g belongs to, where $w = |e|$, so it suffices to check that the same is true of the RHS. The first equation is true because $\pi_e = \pi_{\tilde{\bar{e}}}$.
For the second, it suffices to show that if $h \in \pi_{\tilde{e}}$, then $hg_e^{1_e-1}\tilde{o(e)} = g_e^{1_e-1}\tilde{o(e)}$, i.e.

$$g_e^{1-1_e}\pi_{\tilde{e}}g_e^{1_e-1} \subset \pi_{\tilde{o(e)}} = G_{o(e)}$$

If $1_e = 1, |e| = e$, and $\pi_{\tilde{e}} = G^{\bar{e}}_{\bar{e}} \subset G_{o(e)}$. If $1_e = 0, |e| = \bar{e}$, and $g_e a^e g_e^{-1} = a^{\bar{e}}$ shows $g_e\pi_{\tilde{e}}g_e^{-1} = G^{\bar{e}}_{\bar{e}} \subset G_{o(e)}$. Showing the third equation is valid is a computation analogous to that for the second; alternatively, replacing e with $\bar{e}$ reduces to the previous case. □

This gives an action of π and an isomorphism from $\pi\backslash\tilde{X}$ to Y. If $e \in E(T)$ then $g_e = 1$ so $o(\tilde{e}) = \tilde{o(e)}, t(\tilde{e}) = \tilde{t(e)}$, so the tree T lifts to a tree $\tilde{T}$ in $\tilde{X}$. In algebraic topology, universal covers of graphs are trees, which makes the next result an important one:

Theorem 6.2. *The graph $\tilde{X}$ constructed above is a tree.*

Proof. $\tilde{X}$ is connected: if e is an edge of Y, both its extremities belong to the tree T so one of the extremities of $\tilde{e}$ belongs to $\tilde{T}$. This shows that if W is the smallest subgraph containing all the edges $\tilde{e}$ from edges of Y, then W is connected. Note also that $\pi \cdot W = \tilde{X}$. It suffices to show that there is a generating set S of π such that $W \cup sW$ is connected for all $s \in S$, since by induction on n $W \cup s_1W \cup s_1s_2W \cdots \cup s_1s_2\cdots s_nW$ would be connected for any $s_1, s_2 \cdots s_n \in S \cup S^{-1}$. Then every element g is of the form $s_1s_2\cdots s_n$ so $W \cup gW$ would be connected for any $g \in \pi$.
Take S to be the union of the $G_{\tilde{P}}$ for vertices P of Y and the $\{g_e\}$ for edges e of Y. If s is from a vertex stabiliser, then it fixes that vertex so W, sW share a common vertex. For the g_e, a computation as before show one of the extremities is fixed, so again there is a common vertex and the union must be connected.
$\tilde{X}$ is acyclic, i.e. it contains no closed path of positive length without backtracking: suppose $\tilde{c} = (s_1\tilde{e_1}, \cdots s_n\tilde{e_n})$ is such a path. Let this project to a path c in Y with vertices $(P_0, P_1, \cdots P_n = P_0)$. For ease of notation let 1_i denote 1_{e_i} and $g_i = g_{e_i}$. Then:

$$\begin{aligned}
s_n g_n^{1_n} \tilde{P}_0 &= t(s_n\tilde{e_n}) = o(s_1\tilde{e_1}) = s_1 g_1^{1_1 - 1}\tilde{P}_0 \\
s_1 g_n^{1_1} \tilde{P}_1 &= t(s_1\tilde{e_1}) = o(s_2\tilde{e_2}) = s_2 g_2^{1_2 - 1}\tilde{P}_1 \\
&\cdots \\
s_{n-1} g_{n-1}^{1_{n-1}} \tilde{P_{n-1}} &= t(s_{n-1}\tilde{e_{n-1}}) = o(s_n\tilde{e_n}) = s_n g_n^{1_n - 1}\tilde{P_{n-1}}
\end{aligned}$$

Let $q_i = s_i g_i^{1_i - 1}$. This implies that there exist $r_i \in G_{P_i}$ such that

$$\begin{aligned}
q_1 g_1 r_1 &= q_2 \\
&\cdots \\
q_{n-1} g_{n-1} r_{n-1} &= q_n \\
q_n g_n r_n &= q_1
\end{aligned}$$

Substituting each line into the next line successively gives $g_1r_1 \cdots g_nr_n = 1$. Define $\mu = (1, r_1, r_2 \cdots r_n)$. We will show that (c, μ) is reduced, which gives the desired contradiction (with 5.5).
Suppose $e_{i+1} = \bar{e}_i, g_{i+1} = g_i^{-1}$. This implies $1_{i+1} = 1 - 1_i$. Substituting these into the equation $s_i g_i^{1_i - 1} g_i r_i = s_{i+1} g_{i+1}^{1_{i+1} - 1}$ from above, $r_i \in G_{e_i}^{e_i}$ if and only if $s_i^{-1}s_{i+1} \in g_i^{1_i} G_{e_i}^{e_i} g_i^{-1_i}$. There is no backtracking in $\tilde{X}$ so

$$\overline{s_i\tilde{e_i}} \neq s_{i+1}\tilde{e_{i+1}} = \overline{s_{i+1}\tilde{e_i}}$$

Hence $s_i^{-1}s_{i+1} \notin g_i^{1_i} G_{e_i}^{e_i} g_i^{-1_i}$, so $r_i \notin G_{y_i}^{y_i}$ and the word is reduced. $\square$

Some examples for the reader to test:

1. Suppose all the vertex stabilisers are trivial, so the fundamental group of the graph is the fundamental group in the sense of algebraic topology. Then $\tilde{X}$ is the universal covering (in the sense of topological spaces) of Y relative to T.

2. The construction of $\tilde{X}$ in general closely resembles that done for a segment $Y\ \overset{P}{\circ}\!\xrightarrow{\ y\ }\!\overset{Q}{\circ}$ in a previous section, so it should be no surprise that in the case of a segment, $\tilde{X}$ is the associated tree.

6.2 The Structure Theorem

Finally, we reach the key theorem after much technical (and perhaps at times tedious) preparation. Let a group G act without inversions on a connected graph X. If X is simply-connected, i.e. a tree, we will see that G can be identified with the the fundamental group of a certain graph of groups $(\mathbb{G}, Y)$.
Let T be a maximal tree of the quotient graph $Y = G\backslash X$ and let $j : T \to X$ be a lift. Fix an orientation E_+. We extend this to a section $j : E(Y) \to E(X)$ such that $\overline{j(e)} = j(\bar{e})$; it suffices to define the image of $e \in E_+\backslash E(T)$. Pick $j(e)$ such that $o(j(e))$ is a vertex in the lift of T, so that $o(j(e)) = j(o(e))$. $t(j(e))$ projects to $t(e)$ in $V(Y)$, as does $j(t(e))$, so there is some $\gamma_e \in G$ so that $t(j(e)) = \gamma_e j(t(e))$. This also induces a map from $E_+\backslash E(T) \to G$ that extends to a map $E(Y) \to G$ by the formulae $\gamma_{\bar{e}} = \gamma_e^{-1}$ and $\gamma_e = 1$ if $e \in E(T)$. For each edge $e \in E(Y)$ this gives

$$o(j(e)) = \gamma_e^{1_e - 1} j(o(e))$$
$$t(j(e)) = \gamma_e^{1_e} j(t(e))$$

which, in plain English, says that if the edge is in the orientation, the terminus gets moved, so for the reverse edge not in the orientation, the origin gets moved while the terminus gets fixed.
With vertex and edge stabilisers of a vertex Q and edge z of X defined as G_Q and G_z, define the graph of groups $(\mathbb{G}, Y)$ by $G_P = G_j(P)$ and $G_e = G_{j(e)}$. The injection $G_e \to G_{t(e)}$ is given by $a \mapsto a^e = \gamma_e^{-1_e} a \gamma_e^{1_e}$. This is legitimate since $\gamma_e^{-1_e} G_{j(e)} \gamma_e^{1_e} \subset \gamma_e^{-1_e} G_{t(j(e))} \gamma_e^{1_e} = G_{j(t(e))}$, the last equality coming from the definition of $t(j(e))$.
Let $\phi : \pi_1(\mathbb{G}, Y, T) \to G$ to be the homomorphism defined by the inclusions $G_P \to G$ and $\phi(g_e) = \gamma_e$. Let $\psi : \tilde{X}(\mathbb{G}, Y, T) \to X$ be the map defined by

$$\psi(g\tilde{P}) = \phi(g) j(P)$$
$$\psi(g\tilde{e}) = \phi(g) j(e)$$

It is easy to check that this is a morphism of graphs and is $\phi-$equivariant.
Let W be the smallest subgraph of X containing $j(e)$ for all edges $e \in E(Y)$. Each edge of W has an extremity in $j(T)$ and $G \cdot W = X$. W is contained in $\psi(\tilde{X})$ (taking $g = 1$ above) and ϕ induces isomorphisms between stabilisers of corresponding vertices and edges of $\tilde{X}$ and X. Applying lemma 4.7 gives that ϕ (and hence ψ) is surjective and the fact that ϕ induces isomorphisms between stabilisers shows that ψ is locally injective.

Theorem 6.3. *With the above notation, TFAE:*

i X *is a tree*

ii ψ *is a graph isomorphism*

iii ϕ *is a group isomorphism*

Note how similar the statement is to the corresponding one in the section on fundamental domains. The proof below also shares some similarities.

Proof. (i) $\implies$ (ii): The lemma on locally injective implies injective from the section on fundamental domains applies here to show ψ is injective. Since it is surjective, it is an isomorphism.
(ii) $\implies$ (i): Not so long ago the universal cover $\tilde{X}$ was shown to be a tree.
(ii) $\implies$ (iii): Both morphisms are surjective so it suffices to check injectivity. We prove the contrapositive. Let N be the kernel of ϕ and $P \in V(Y)$. $N \cap G_{\tilde{P}}$ is trivial since ϕ defines an isomorphism between stabilisers. If n is a non-trivial elemnt of N, then it can't fix a vertex of Y so $n\tilde{P}, \tilde{P}$ are distinct vertices in $\tilde{X}$ with the same image $j(P)$ in X, so ψ not injective.
(iii) $\implies$ (ii): If G is isomorphic to the fundamental group, then the action of π_1 on X has all the defining properties of the universal cover, so X is the universal cover. $\square$

The result we have been aiming towards has now been proved: (i) $\implies$ (iii) is precisely the statement that if a group acts without inversions on a tree X, then $\pi_1(\mathbb{G}, G\backslash X) \simeq G$.

Corollary 6.4. *Let X be a tree. Let R be the subgroup of G generated by the $G_P, P \in V(X)$. Then R is a normal subgroup of G and G/R can be identified with the fundamental group (in the sense of topological spaces) of the graph $Y = G\backslash X$.*

Proof. Once G is identified with $\pi_1(\mathbb{G}, Y, T)$, this follows from the remark made after we proved the fundamental group is well-defined. $\square$

A meatier consequence of the structure theorem is as below. Suppose $H = *_A H_i$ is an amalgam of groups $(H_i)_{i \in I}$ along a common subgroup A. Let G be a subgroup of H, and for each *coset* $x \in H/h_i$ let $G_{i,x} = G \cap xH_ix^{-1}$. $G_{i,x}$ can then be seen as the stabiliser of x under the action of G on H/H_i by left multiplication. Then we have

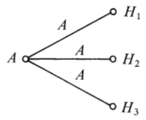
Theorem 6.5. *Suppose $G\backslash\{1\}$ doesn't meet any conjugate of A. Then there exists*

i a free subgroup F of G

ii for each $i \in I$ a subset X_i of H/H_i which is a system of coset representatives for $G\backslash H/H_i$

such that $G = (*_{i \in I, x \in X_i} G_{i,x}) * F$.

In plain(er) English, G is the free product of its intersections with the conjugates of H_i and a free group.



Proof. Define a tree of groups $(\mathbb{H}, T_0)$ having a vertex A which is joined to vertices H_i and has no other edges. The associated groups are the obvious ones. Give each edge the group A with the natural injection into its extremities. Then $H = \varinjlim(\mathbb{H}, T_0)$. Let X be the tree associated to this tree of groups. H acts on X with T as fundamental domain in such a way that stabilisers of edges are conjugates of A and stabilisers of vertices are conjugates of either A or the H_i. G can then be made to act on X by restricting the action from H. Define $Y = G\backslash X$ and let T be a maximal tree of Y. By the structure theorem, $G \simeq \pi_1(\mathbb{G}, Y, T)$. The hypothesis about not meeting conjugates of A means that the stabiliser in G of each edge of Y is trivial. Then there is a free group F such that

$$\pi_1 \simeq (*_{P\in V(T)} G_P) * F$$

(Recall that the fundamental group can be built up by just gluing more groups as they come along. Here only the identity has to be glued.) Note that by comparing with the second remark made after we proved that the fundamental group is well defined, F is the fundamental group (in the sense of topological spaces) $\pi_1(Y, T)$. The construction of X implies $V(X) \simeq H/A \bigsqcup_{i\in I} H/H_i$. After quotient by the $G-$ action,

$$V(T) \simeq G\backslash H/A \bigsqcup G\backslash H/H_i$$

and T can be lifted into X to give a system of representatives

$$X_A \subset H/A, X_i \subset H/H_i$$

of $G\backslash H/A, G\backslash H/H_i$ respectively. For $x \in X_i$, the corresponding group G_P is $G \cap x H_i x^{-1}$. The same applies for $x \in A$, but $G \cap x A x^{-1}$ is trivial, so substituting these into the free product for π_1 gives the theorem. $\square$

It would be pretty silly if it turns out the hypotheses of the theorem are never satisfied. Luckily they are at least in the trivial case $A = \{1\}$, from which one obtains a result known as the Kurosh subgroup theorem. It took Alexander Kurosh 14 pages to prove this in his 1934 paper [2], which might suggest to the reader how powerful the theory that lies beyond is. This is just the beginning.

References

[1] Jean-Pierre Serre, J. Stilwell - Trees (1980, Springer)

[2] Alexander Kurosh, Die Untergruppen der freien Produkte von beliebigen Gruppen. Mathematische Annalen, vol. 109 (1934), pp. 647–660.