# Property (T) and (FA) for random Groups in the $l$-angular model

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**Abstract**

In this paper we prove some results, in particular about Serre's Property (FA) and Kazhdan's Property (T), for the $l$-angular model of random groups that are known for the Gromov model. The method of showing certain groups don't have Property (T) follows that of Ollivier and Wise, constructing walls in the Cayley complex, which has subsequently been used by other authors in [MP15] and [Mon21] with improved bounds.

## 1 Introduction

The Gromov model for random groups refers to the following method of generating a random group: fix a set of generators $g_1 \dots g_n$ and choose at random a set of $(2n-1)^{dl}$ reduced words of length $l$. For various properties of groups, such as hyperbolicity, small cancellation, Property (T) and so on, one can calculate the probability it holds in a random group. See [Oll05] for a survey of important results and open questions. In particular, as $l \to \infty$ one obtains some indication of what properties a 'generic' group at density $d$ has (this should be taken with the caveat that there is no way of putting a uniform probability measure on the set of all countable groups). If the limiting probability is 1, the property is said to hold with overwhelming probability (w.o.p.). For example, when Gromov first introduced his model of random groups in [Gro93] he proved that at density $d < \frac{1}{2}$ a random group is hyperbolic w.o.p., while if $d > \frac{1}{2}$ a random group is $\mathbb{Z}/(2)$ or trivial w.o.p.
Given the ubiquity of Property (T) in mathematics, it is natural to consider at what densities a random group has Property (T). For the Gromov model, this is quite difficult as the lengths of relators tend to infinity. It is much easier to fix a length of relators, and consider the behaviour as the number of generators tends to infinity, which is what is done here.

**Definition 1** ($l-$angular model for random groups)**.** Fix a natural number $l$ and a *density* $d \in (0, 1)$. Let $W_n$ be the set of all cyclically reduced words of length $l$ over $S_n$, an alphabet of size $n$, allowing inverses of letters. Let $R_n$ be a subset of $W_n$ having $(2n-1)^{ld}$ elements, chosen at random with the uniform distribution among all such sets. We define a random group in the $\mathbb{G}(n, l, d)$

model to be a group given by the presentation $\langle S_n | R_n \rangle$. The model $\mathbb{G}(n, l, d)$ will be also called the $l-$gonal model on $n$ generators at density $d$.

Note that this is also known as the $k$-angular model, where $k$ plays the same role as $l$. When $l = 3$, the model is called the triangular model or the Zuk model. More is known about Property (T) in the $l$-angular model than in the Gromov model (see [Odr]).
In section 2 we prove that a random group in the $l$-angular model has the small cancellation property and at low densities has the Greendlinger property and hence solvable word problem. In section 3 we prove that at low densities does not have Property (T) and give a range of densities for which they have Property (FA). The main results are:

**Theorem** (proposition 12)**.** For $l \geq 7$ and any $\frac{3}{l} < d < \frac{1}{2}$, a random group in the $l$-angular model satisfies property (FA).

**Theorem** (theorem 15)**.** When $d < \frac{1}{5}$ w.o.p. a random group at density $d$ doesn't have Property (T).

This is done by finding a codimension-1 wall in the Cayley complex, whose stabiliser is a codimension 1 subgroup, and this gives rise to an "essential" action of the group on a CAT(0) cube complex by the work of Sageev in [Sag95]. We note that the arguments in [OW09] apply verbatim to show that in fact a random group at density $d < \frac{1}{6}$ acts freely and cocompactly on a CAT(0) cube complex.

# 2 Small Cancellation

Small cancellation was first developed in [Gr60] and has subsequently become an extremely useful tool for constructing groups with exotic properties, for example Olshanskii's construction of Tarski monsters [Ol80]. We show in this section that, analogous to the Gromov model, random groups in the $l$-angular model satisfy a small cancellation condition.

**Definition 2.** Let $G = \langle S | R \rangle$ be a group presentation where $R \subseteq F(X)$ is a set of freely reduced and cyclically reduced words in the free group $F(X)$ such that $R$ is *symmetrized*, that is, closed under taking cyclic permutations and inverses.

A nontrivial freely reduced word $u$ in $F(X)$ is called a *piece* with respect to the presentation if there exist two distinct elements $r_1$, $r_2$ in $R$ that have $u$ as maximal common initial segment.

Note that any presentation can be symmetrized by just adding in the cyclic permutations and inverses without altering the isomorphism type of the group.

**Definition 3.** Let $0 < \lambda < 1$. The presentation is said to satisfy the $C'(\lambda)$ small cancellation condition if whenever $u$ is a piece w.r.t. the presentation and $u$ is a subword of some $r \in R$, then $|u| < \lambda |r|$. Here $|v|$ is the length of a word $v$.

This is useful because it allows one to do induction on van Kampen diagrams by successively removing discs.

**Definition 4.** A *(singular) disc diagram* is a compact, contractible subcomplex of $\mathbb{R}^2$. If we orient $\mathbb{R}^2$, then $D$ has a well-defined boundary cycle. A disc diagram is *labelled* if each (oriented) edge is labelled by an element $s \in S^{\pm}$. The set of *face labels* is the set of boundary cycles of the 2-cells. If all the face labels are elements of $R^{\pm}$, then we say $D$ is a *diagram* over $\langle S \mid R \rangle$.

**Definition 5** (van Kampen diagrams)**.** If $w \in \langle\langle R \rangle\rangle$, and $w$ is the boundary cycle of a singular disc diagram $D$ over the presentation $\langle S|R \rangle$, we say that $D$ is a *van Kampen diagram* for $w$.

If the van Kampen diagram has a disc with long boundary word, then removing the disc leaves another van Kampen diagram with fewer discs but with similar length to area ratio and this is what allows us to do induction.
In [Gro93] Gromov gives explains reason why random groups in the Gromov model should satisfy the small cancellation property. Below we give a concrete calculation for the $l$-angular model.

**Proposition 6.** Let $d < \frac{1}{2}$. Fix $\epsilon > 0$. With overwhelming probability, a random group in the $l$-angular model at density $d$ satisfies the $C'(2d + \epsilon)$ small cancellation condition but doesn't satisfy the $C'(2d - \epsilon)$ condition.

*Proof.* Note that there are approximately $(2n - 1)^l$ cyclically reduced words of length $l$ in $F_n$, so for a fixed word $w$ of length $\lambda l$, $0 < \lambda < 1$, there are $l(2n-1)^{l-\lambda l}$ words containing $w$ as subword ($l$ accounts for the cyclicity). Hence there are $\binom{l(2n-1)^{l-\lambda l}}{2}$ pairs of words which, if chosen as relators, would violate the $C'(\lambda)$ condition by having $w$ as a subword, and there are $(2n - 1)^{\lambda l}$ choices for $w$.
Let $|v|$ denote the length of a word $v$, $w \in v$ mean $w$ is a subword of $v$, and $1_{v_1,v_2}$ be the indicator function for $v_1, v_2$ both being chosen as relators. Note that for any word $w$ satisfying $w \in v_1, w \in v_2$,

$$\mathbb{E}[1_{v_1,v_2}] = \frac{\binom{(2n-1)^l-2}{(2n-1)^{l\lambda}-2}}{\binom{(2n-1)^l}{(2n-1)^{l\lambda}}}.$$

Consider $X = \sum_{|w|=\lambda} \sum_{w\in v_1, w\in v_2} 1_{v_1,v_2}$. Then $C'(\lambda)$ is satisfied if and only if $X = 0$.
Suppose first $\lambda = 2d + \epsilon$. Using linearity of expectation,

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{|w|=2d+\epsilon} \binom{l(2n-1)^{l-(2d+\epsilon)l}}{2} \frac{\binom{(2n-1)^l-2}{(2n-1)^{ld}-2}}{\binom{(2n-1)^l}{(2n-1)^{ld}}} \\
&= (2n-1)^{(2d+\epsilon)l} \binom{l(2n-1)^{l-(2d+\epsilon)l}}{2} \frac{\binom{(2n-1)^l-2}{(2n-1)^{ld}-2}}{\binom{(2n-1)^l}{(2n-1)^{ld}}} \\
&\leq (2n-1)^{(2d+\epsilon)l} (l(2n-1)^{l-(2d+\epsilon)l})^2 \frac{\binom{(2n-1)^l-2}{(2n-1)^{ld}-2}}{\binom{(2n-1)^l}{(2n-1)^{ld}}}
\end{aligned}$$

Expanding the binomial coefficient,

$$\frac{\binom{(2n-1)^l}{(2n-1)^{ld)}}}{\binom{(2n-1)^l-2}{(2n-1)^{ld}-2}} = \frac{(2n-1)^{ld}((2n-1)^{ld)}-1)}{(2n-1)^l((2n-1)^l-1)} \approx (2n-1)^{-2l(1-d)}.$$

Hence the expression for $\mathbb{E}[X]$ is

$$O((2n-1)^{l(2d+\epsilon-2(1-(2d+\epsilon))-2(1-d)}) = O((2n-1)^{-\epsilon})$$

which tends to 0 as $n \to \infty$. By Markov's inequality, $\mathbb{P}(X=0) \to 1$ as $n \to \infty$. Now suppose $\lambda = 2d - \epsilon$. Note $1_{v_1,v_2} 1_{v_3,v_4} = 1_{v_1,v_2,v_3,v_4}$, where $\{v_1, v_2, v_3, v_4\}$ is treated as a multiset. Then

$$\begin{aligned} X^2 = & \sum_{|w|=2d-\epsilon} \sum_{w \in v_1, w \in v_2} 1^2_{v_1,v_2} \\ & + 2 \sum_{|w|=2d-\epsilon} \sum_{w\in v_1, w \in v_2, w \in v_3, w \in v_4} 1_{v_1,v_2,v_3,v_4} \\ & + 2 \sum_{|w_1|=2d-\epsilon=|w_2|, w_1 \neq w_2} \sum_{w_1, w_2 \in v_1, v_2} 1_{v_1,v_2} \\ & + 2 \sum_{|w|=2d-\epsilon} \sum_{w \in v_1,v_2,v_3} 1_{v_1,v_2,v_3} \\ & + 2 \sum_{|w_1|=2d-\epsilon=|w_2|, w_1 \neq w_2} \sum_{w_1 \in v_1,v_2, w_2 \in v_2,v_3} 1_{v_1,v_2,v_3} \\ & + 2 \sum_{|w_1|=2d-\epsilon=|w_2|, w_1 \neq w_2} \sum_{w_1 \in v_1,v_2, w_2 \in v_3,v_4} 1_{v_1,v_2,v_3,v_4} \end{aligned}$$

The first term is just $X$, so has expectation $O((2n-1)^{l\epsilon})$ by a similar calculation as above.
The second term has expectation

$$\begin{aligned} & 2(2n-1)^{l(2d-\epsilon)} \binom{l(2n-1)^{l(1-2d+\epsilon)}}{4} \mathbb{E}[1_{v_1,v_2,v_3,v_4}] \\ & = 2(2n-1)^{l(2d-\epsilon)} \binom{l(2n-1)^{l(1-2d+\epsilon)}}{4} \frac{\binom{(2n-1)^l-4}{(2n-1)^{ld}-4}}{\binom{(2n-1)^l}{(2n-1)^{ld}}} = O((2n-1)^{l(3\epsilon-2d}). \end{aligned}$$

For the third term, we split further based on whether $|w_1 \cap w_2| = 0$ or not. If $|w_1 \cap w_2| = i > 0$, then the number of $v_i$ containing both words concatenated is $l(2n-1)^{l(1-2(2d-\epsilon))+i}$ and since either $w_1$ precedes $w_2$ or vice versa, there are $2(2n-1)^{2l(2d-\epsilon)-i}$ pairs $(w_1, w_2)$, so the contribution to expected value from such pairs is $lO((2n-1)^{l(2\epsilon-d)+i})$ (note $l$ is fixed)
If $|w_1 \cap w_2| = 0$, then the number of pairs of subwords is $\binom{(2n-1)^{l(2d-\epsilon)}}{2}$ and for each pair of subwords there are at most $\binom{l^2(2n-1)^{l(1-2(2d-\epsilon))}}{2}$ pairs of words

containing $w_1, w_2$ as disjoint subwords. Hence the contribution to expectation is$O((2n-1)^{2l(\epsilon-d)})$.
For the fourth term, a computation similar to that for the second term gives an expectation $O((2n-1)^{l(2\epsilon-d})$.
The calculation for the fifth term is similar to that of the third term. Split by the intersection of $w_1, w_2$ in $v_2$.
If $|w_1 \cap w_2| = i > 0$, then the number of $v_i$ containing both words concatenated is $l(2n-1)^{l(1-2d+\epsilon)-i}$ and since either $w_1$ precedes $w_2$ or vice versa, there are $2(2n-1)^{l(2d-\epsilon)+i}$ such pairs $(w_1, w_2)$. There are then $(2n-1)^{l(1-2(2d-\epsilon))+i}$ choices for $v_2$, and at most $(2n-1)^{l(1-2d+\epsilon)} - 1$ choices for each of $v_1, v_3$, so the contribution to expected value is $lO((2n-1)^{l(3\epsilon-2d)+i})$.
If $|w_1 \cap w_2| = 0$, then the number of $v_i$ containing both words concatenated is $l(2n-1)^{l(1-4d+2\epsilon)}$ and there are $\binom{(2n-1)^{l(2d-\epsilon)}}{2}$ such pairs $(w_1, w_2)$. There are then $(2n-1)^{l(1-2(2d-\epsilon))}$ choices for $v_2$, and at most $(2n-1)^{l(1-2d+\epsilon)} - 1$ choices for each of $v_1, v_3$, so the contribution to expected value is $O((2n-1)^{l(2\epsilon-d)})$.
The sixth term has expectation

$$2\binom{(2n-1)^{l(2d-\epsilon)}}{2}\binom{l(2n-1)^{l(1-2d+\epsilon)}}{2}^2 \frac{\binom{(2n-1)^l-4}{(2n-1)^{ld}-4}}{\binom{(2n-1)^l}{(2n-1)^{ld}}} = \frac{l^2}{4}(2n-1)^{2l\epsilon} + o((2n-1)^{2l\epsilon}).$$

Compare with $\mathbb{E}[X]^2 = \frac{l^2}{4}(2n-1)^{2l\epsilon} + o((2n-1)^{2l\epsilon})$. Note in both terms 3 and 5, $i < ld$, which means that $(2n-1)^{2l\epsilon}$ is asymptotically larger than any of the terms apart from term 6, and the coefficients agree so $\frac{\mathbb{E}[x]^2}{\mathbb{E}[X^2]} \to 1$ so by the second moment method $\mathbb{P}(X=0) \to 0$. □

We now show that the Greendlinger property holds at low densities, allowing the word problem to be solved by the Dehn algorithm in these cases. The following lemmas are a form of isoperimetric inequality for van Kampen diagrams that will be useful throughout the rest of this paper.

**Lemma 7.** (Theorem 2 in [Oll07], Theorem 2.6 in [Odr] for the $l-$angular model.) At density $d$, for any $\epsilon > 0$ the following property occurs with overwhelming probability: all reduced van Kampen diagrams $D$ satisfy

$$|\partial D| > (1 - 2d - \epsilon)l|D|$$

**Lemma 8.** For any $\epsilon > 0$ , with overwhelming probability, at density $d$ the following holds: Let $D$ be a reduced van Kampen diagram with at least two faces. There exist two faces of $D$ each having at least $l(1-5d/2-\epsilon)$ consecutive edges on the boundary of $D$.

*Proof.* With lemma 7, the proof of Lemma 16 (the lemma statement but without 'consecutive') and the subsequent proof of the Greendlinger property in [Oll07] goes through, so is true in the $l$-angular model too.

**Proposition 9.** A random group in the $l$-angular model satisfies the Greendlinger property whenever $d < 1/5$ for $l$ even, or $d < \frac{1}{5} + \frac{1}{5l}$ for $l$ odd.

*Proof.* When $d < \frac{1}{5}, l(1-5d/2-\epsilon) > \frac{l}{2}$. When $l$ is even, this is optimal as there can be words of length exactly $l/2$. However, when $l$ is odd we only need $l(1-5d/2-\epsilon) > \frac{l-1}{2}$, which is possible for $d < \frac{1}{5} + \frac{1}{5l}$ if $\epsilon$ is chosen to be small enough. □

We also note that Ollivier's proof of the failure of the Dehn algorithm also goes through as given in his paper provided, in our case, that $l$ is large enough. His construction requires $(d-\epsilon)l, (2d-\epsilon)l, (d-\epsilon)l/2$ to be lengths of subwords, so in particular integers. At the end of his paper, he derives a contradiction from $7(1-2d-\epsilon')l < 6(1-2d)l + 8\epsilon l - 2$ for sufficiently small $\epsilon, \epsilon'$. In our case, $\epsilon, \epsilon'$ are subject to the same restrictions, so we may take them to be equal. But there is some $\epsilon \in (0, \frac{2}{l})$ making these integers.

$$7(1-2d-\epsilon)l \geq 6(1-2d)l + 8\epsilon l - 2 \qquad \Leftrightarrow 15\epsilon + 2d \leq 1 + 2/l \qquad (1)$$

$$1 - 5d/2 + \frac{3\epsilon}{2} < \frac{1}{2} \qquad \Leftrightarrow 1 + 3\epsilon < 5d \qquad (2)$$

Eliminating $d$, we require $\epsilon < \frac{5}{81}(\frac{1}{2} + \frac{1}{l})$. When the RHS is greater than $\frac{2}{l}$, this is certainly possible. Solving for $l$, this is equivalent to $l \geq 63$.

# 3 Property (FA) and Property (T)

**Definition 10** (Definition 2.1 in [DGP])**.** A *basic automaton* (also abbreviated as a *b-automaton*) over an alphabet $S$ with transition data $\{\sigma_s\}$ is a pair $(S, \{\sigma_s\})$, where $\{\sigma_s\}_{s \in \{\varnothing\} \cup S^{\pm}}$ is a family of subsets of $S^{\pm}$.
The *language* of a b-automaton with transition data $\{\sigma_s\}$ is the set of all (nonempty) words over $S$ beginning with a letter in $\sigma_\varnothing$ and such that for any two consecutive letters $ss'$ we have $s' \in \{\sigma_s\}$. We say that a b-automaton is $\lambda-$large, for some $\lambda \in (0,1)$, if $\sigma_\varnothing \neq \varnothing$ and for each $s \in S^{\pm}$ we have $|\sigma_s| \geq \lambda 2n$.

**Definition 11.** Let $G$ be a group acting without inversions on a tree $X$. Denote by $X^G$ the set of points fixed by $G$. $G$ is said to satisfy (FA) if $X^G \neq \varnothing$ for any tree $X$ that $G$ acts on.

**Proposition 12.** For $l \geq 7$ and any $\frac{3}{l} < d < \frac{1}{2}$, a random group in the $l$-angular model satisfies property (FA).

*Proof.* Following the notation and arguments in [DGP], it suffices to prove that a random set of relators intersects all the $\frac{1}{3}$-large automata. There are $2^{2n(2n+1)}$ automata in total, and if an automata is $\frac{1}{3}$-large then it has at least $(\frac{2n}{3}-1)^{l-1}$ words of length $l$, so for a fixed automata, the probability a set of random relators doesn't intersect it is $\frac{\binom{(2n-1)^l - (\frac{2n}{3}-1)^{l-1}}{(2n-1)^{dl}}}{\binom{(2n-1)^l}{(2n-1)^{dl}}}$.

Applying Stirling's approximation and regrouping terms gives

$$\sqrt{\frac{((2n-1)^l - (\frac{2n}{3}-1)^l)((2n-1)^l - (2n-1)^{dl})}{(2n-1)^l((2n-1)^l - (\frac{2n}{3}-1)^l - (2n-1)^{dl})}}$$
$$\times (1 + \frac{(2n-1)^{dl}}{(2n-1)^l - (2n/3-1)^{l-1} - (2n-1)^{dl}})^{(2n-1)^l - (2n/3-1)^{l-1}}$$
$$\times (1 - (2n-1)^{-(1-d)l})^{(2n-1)^l}$$
$$\times (1 - \frac{(\frac{2n}{3}-1)^{l-1}}{(2n-1)^l - (2n-1)^{dl} - \frac{2n}{3} - 1)^{l-1}})^{(2n-1)^{dl}})$$

The expression in the square root is asymptotically 1 so it suffices to consider the remaining terms
Since $1 + x \leq e^x$ for all real $x$, applying this to each bracket in the expression shows this expression is $\leq$

$$\exp((2n-1)^{dl}$$
$$\times (\frac{(2n-1)^{dl}}{(2n-1)^l - (2n-1)^{dl} - \frac{2n}{3} - 1)^{l-1}} - 1 - \frac{\frac{2n}{3} - 1)^{l-1}}{(2n-1)^l - (2n-1)^{dl} - \frac{2n}{3} - 1)^{l-1}})$$
$$\times \frac{(2n-1)^{dl}}{(2n-1)^l - (2n-1)^{dl} - \frac{2n}{3} - 1)^{l-1}})$$

which tends to 0, and

$$(\frac{(\frac{2n}{3}-1)^{l-1}}{((2n-1)^l - (2n-1)^{dl} - \frac{2n}{3} - 1)^{l-1}}) = O(n^{-1})$$

Letting $X = \sum_{u \text{ is a } \frac{1}{3}\text{-large automata}} 1_{u \text{ missed}}$,

$$\mathbb{E}[X] \leq e^{-O(n^{dl-1})} 2^{2n(2n+1)} \to 0$$

since $dl - 1 > 2$ so the probability of any $\frac{1}{3}$-large automata being missed tends to zero. □

**Remark.** When $l = 2$, there are $2n(2n-1)$ reduced words of length 2 and, for a fixed generator, at most $4(2n-1)$ words involving that generator. Hence there is a high probability that the chosen relators don't involve some generator, meaning the group $G$ splits as a free product $\mathbb{Z} \star H$ (where $H$ is some quotient of $F_{n-1}$), so $G$ doesn't have FA [Se80]. When $d < \frac{1}{l}$, a random group at density $d$ is free with overwhelming probability, so won't have FA. [AR17]

We now discuss Property (T). The standard reference is [BdlHV].

**Definition 13.** Let $(\pi, \mathbb{H})$ be a continuous unitary representation of a locally compact, $\sigma-$compact topological group $G$.

1. Given a subset $S \subseteq G$ and a number $\epsilon > 0$, a unit vector $v$ in $H$ is $(S, \epsilon)$-invariant if $sup_{g\in S}||\pi(g)v - v|| \leq \epsilon||v||$.

2. The representation $(\pi, \mathbb{H})$ has almost invariant vectors if it has $(K, \epsilon)$–invariant vectors for every compact subset $K$ of $G$ and every $\epsilon > 0$.

3. The representation $(\pi, \mathbb{H})$ has invariant vectors if there exists a unit vector $v$ in $\mathbb{H}$ such that $\pi(g)v = v$ for all $g \in G$.

**Definition 14.** A locally compact Hausdorff topological group $G$ has Kazhdan's Property (T) if for every unitary representation $\pi$ of $G$, if $\pi$ has almost invariant vectors then it also has invariant vectors.

Property (T) implies Property (FA) [Wat81], but comparing the next two propositions gives a range of densities of random groups where the converse is not true.

**Theorem 15.** When $d < \frac{1}{5}$ w.o.p. a random group at density $d$ doesn't have Property (T).

We follow the proof strategy used in [OW09].
Define a *hypergraph* on a Cayley complex to be a graph $\Gamma$ whose set of vertices is the set of 1-cells in the Cayley complex. There is an edge in $\Gamma$ between two vertices if there is some 2-cell $R$ in the Cayley complex such that these vertices correspond to antipodal 1-cells in the boundary of $R$. $R$ is the 2-cell *containing* the edge. If $l$ is odd subdivide all 1-cells of $\tilde{X}$ so that hypergraphs can be defined.

**Lemma 16** (Lemma 2.3 in [OW09])**.** Suppose a hypergraph $\Lambda$ is an embedded tree in the simply connected Cayley complex $\tilde{X}$. Then $\tilde{X} - \Lambda$ consists of two components.

*Proof.* Since $\tilde{X}$ is simply connected, $H_1(\tilde{X}) = 0$. Let $U$ be a small neighbourhood of the tree and $V$ the complement of the tree. Note that one can take $U$ to be homeomorphic to a small thickening of the tree, say $\Lambda \times (-\epsilon, \epsilon)$, since all the 2-cells are disjoint from each other, so within a 2-cell the neighbourhood of the tree is just a neighbourhood of the diameter of a disc. These can then be taken so that they are glued together compatibly. Then $U \cap V = \Lambda \times (-\epsilon, \epsilon) - \Lambda \times \{0\}$. The homological Mayer-Vietoris sequence gives:

$$\begin{aligned} H_1(\tilde{X}) = 0 \to H_0(U \cap V) &= H_0(\Lambda \times (-\epsilon, \epsilon) - \Lambda \times \{0\}) \\ &\to H_0(\tilde{X} - \Lambda) \oplus H_0(\Lambda \times (-\epsilon, \epsilon)) \to H_0(\tilde{X}) \to 0 \end{aligned}$$

Since $U \cap V$ is homeomorphic to $\Lambda \times (-\epsilon, 0) \cup \Lambda \times (0, \epsilon)$, which deformation retracts to two copies of $\Lambda$, the sequence is then

$$0 \to \mathbb{Z}^2 \to H_0(\tilde{X} - \Lambda) \oplus \mathbb{Z} \to \mathbb{Z} \to 0$$

Hence $H_0(\tilde{X} - \Lambda) \cong \mathbb{Z}^2$. $\square$

**Proposition 17** (Lemma 7.3 in [OW09])**.** When $l$ is odd and $d < \frac{1}{5}$ or when $l$ is even and $d < \frac{1}{5} + \frac{1}{5l}$, with overwhelming probability the hypergraphs of the Cayley 2-complex of a random group at density $d$ are embedded leafless trees.

*Proof.* Note that proposition 9 implies that at the given densities, the following condition (Condition 4.2. in [OW09]) is satisfied with overwhelming probability: For every reduced spurless van Kampen diagram $D \to X$ either
(1) $D$ has at most one 2-cell
(2) $D$ contains at least two shells.
Following the arguments given in [OW09] show that the hypergraph is an embedded tree at the stated densities. To show that it is leafless, it suffices to show that every generator is contained in at least two relators. (When $l$ is odd the edges have to be subdivided to carry out this construction, but the same argument is valid since if $s_1$ appears in two different relators, every half segment of $s_1$ also appears twice to give the corresponding vertex in the wall at least two neighbours.) Fix a generator $s_1$.
There are $(2n-3)^l$ reduced words of length $l$ not involving $s_1$ (or $s_1^{-1}$), so the probability of picking a set of relators with at most one containing $s_1$ is $\frac{\binom{(2n-3)^l}{(2n-1)^{dl}}}{\binom{(2n-1)^l}{(2n-1)^{dl}}} + \frac{((2n-1)^l-(2n-3)^l)\binom{(2n-3)^l}{(2n-1)^{dl}-1}}{\binom{(2n-1)^l}{(2n-1)^{dl}}}$ $1_i$ be the indicator function for the $i$th generator appearing at most once, $1 \leq i \leq n$, and let $X = \sum_{i=1}^n 1_i$. It suffices to show $\mathbb{E}[X] \to 0$. For large enough $n$, $\frac{((2n-1)^l-(2n-3)^l)\binom{(2n-3)^l}{(2n-1)^{dl}-1}}{\binom{(2n-1)^l}{(2n-1)^{dl}}} \leq \frac{((2n-1)^l-(2n-3)^l)\binom{(2n-3)^l}{(2n-1)^{dl}}}{\binom{(2n-1)^l}{(2n-1)^{dl}}}$ as $(2n-1)^{dl} < \frac{(2n-3)^l}{2}$.
Applying Stirling, $1 + x \leq e^x$, and noting that the square root term is asymptotically one as before,

$$\begin{aligned}
\frac{\binom{(2n-3)^l}{(2n-1)^{dl}}}{\binom{(2n-1)^l}{(2n-1)^{dl}}} &\sim \frac{(2n-3)^{l(2n-3)^l}((2n-1)^l-(2n-1)^{dl})^{(2n-1)^l-(2n-1)^{dl}}}{(2n-1)^{l(2n-1)^l}((2n-3)^l-(2n-1)^{dl})^{(2n-3)^l-(2n-1)^{dl}}} \\
&= (1+\frac{(2n-1)^{dl}}{(2n-3)^l-(2n-1)^{dl}})^{(2n-3)^l}(1-\frac{(2n-1)^{dl}}{(2n-1)^l})^{(2n-1)^l} \\
&\times (1-\frac{(2n-1)^l-(2n-3)^l}{(2n-1)^l-(2n-1)^{dl}})^{(2n-1)^{dl}} \\
&\leq \exp((2n-1)^{dl}(\frac{(2n-3)^l}{(2n-3)^l-(2n-1)^{dl}} - 1 - \frac{(2n-1)^l-(2n-3)^l}{(2n-1)^l-(2n-1)^{dl}}))
\end{aligned}$$

$\frac{(2n-3)^l}{(2n-3)^l-(2n-1)^{dl}} - 1 = \frac{(2n-1)^{dl}}{(2n-3)^l-(2n-1)^{dl}} \to 0$, while $\frac{(2n-1)^l-(2n-3)^l}{(2n-1)^l-(2n-1)^{dl}} \to 1$, so for large $n$, $\frac{\binom{(2n-3)^l}{(2n-1)^{dl}}}{\binom{(2n-1)^l}{(2n-1)^{dl}}} \leq \exp -\frac{(2n-1)^{dl}}{2}$ and hence
$\mathbb{E}[X] \leq n(1+((2n-1)^l-(2n-3)^l)\exp -\frac{(2n-1)^{dl}}{2} \to 0.$ $\square$

**Definition 18.** A disc diagram $D \to \tilde{X}$ is said to be *collared* if it has the following properties

1. there is an external 2-cell $C$ called a corner of $D$
2. there is a hypergraph segment $\lambda \to D \to \tilde{X}$ of length at least 2
3. the first and last edge of $\lambda$ lie in $C$, and no other edge lies in $C$
4. $\lambda$ passes through every other external 2-cell of $D$ exactly once
5. $\lambda$ does not pass through any internal 2-cell of $D$.

**Definition 19.** A cancellable pair in $Y \to X$ is a pair of distinct 2-cells $R_1$, $R_2$ meeting along an edge $e$ in $Y$ such that $R_1$ and $R_2$ map to the same 2-cell in $X$, and moreover, the boundary paths of $R_1$ and $R_2$ starting at $e$, map to the same path in $X$. A map $Y \to X$ is reduced if $Y$ contains no cancellable pairs. Note that the composition of reduced maps is reduced.

**Lemma 20** (Lemma 3.17 in [OW09])**.** Let $\Lambda$ be a hypergraph which is an embedded tree in $\tilde{X}$. Let $\lambda$ be a segment of $\Lambda$. Let $\gamma$ be an embedded path in $\tilde{X}$ with the same endpoints as $\lambda$. (Here $\gamma$ is an edge path which starts and ends at "midedge vertices" corresponding to vertices of $\Lambda$.) Then there exists a reduced diagram $F$ quasicollared by $\lambda$ and $\gamma$. Moreover, in the case $\gamma$ does not intersect $\Lambda$ anywhere except at its endpoints, then $F$ is actually collared.

This lemma, together with lemma 7, is used to prove the next theorem that the hypergraph is quasi-isometrically embedded.

**Theorem 21** (Theorem 4.5 in [OW09])**.** When $l$ is odd and $d < \frac{1}{5}$ or when $l$ is even and $d < \min\{\frac{1}{5} + \frac{1}{5l}, \frac{1}{4}\}$, with overwhelming probability the distance in $\tilde{X^{(1)}}$ between two vertices of a hypergraph $\Lambda$ is at least $(\frac{1}{2} - 2d - \epsilon)l$ times the minimal number of edges joining them in $\Lambda$

**Corollary 22** (Corollary 4.6 in [OW09])**.** In random groups at density $d < \frac{1}{5}$, with overwhelming probability, the stabiliser of any hypergraph is a free, quasiconvex subgroup that acts cocompactly on the hypergraph.

*Proof.* The proof that the stabiliser is free and quasiconvex is identical to in [OW09]. To show that the quotient is compact, note that if two vertices of the hypergraph come from $1-$cells $g_1, g_1 s$ and $g_2, g_2 s$ for the same generator $s$, then the action of $g_1 g_2^{-1}$ shows that they are in the same orbit of the action of the stabiliser, so the number of vertices in the quotient is at most the number of generators. Similarly, if two edges come from $2-$cells with the same boundary word, then they are in the same orbit, so the number of edges in the quotient is at most the number of relations, which is finite. $\square$

**Lemma 23** (Lemma 6.1 in [OW09])**.** Consider a random group at density $d$. Then with overwhelming probability the following holds. Let $D$ be a reduced diagram with $|D| = 3$, and suppose $D$ contains a 2-cell corresponding to relator $r_1$. Then the number of internal 1-cells in $D$ is at most $2dl + \epsilon l$.

The key estimate used is that

$$3(2n-1)^{-dl}\mathbb{E}S_n(D) \leq 3(2n-1)^{2dl-L}$$

and the last term $(2n-1)^{2dl-L}$ tends to zero as $n \to \infty$ if $L \geq 2dl + \epsilon l$. The same observation underlies the proof of lemmas 6.2 and 6.3 in [OW09].

**Theorem 24** (Theorem 7.4 in [OW09])**.** With overwhelming probability, random groups $G$ at density $d < \frac{1}{5}$ have a subgroup $H$ which is free, quasiconvex, and such that the relative number of ends $e(G, H)$ is at least 2. This subgroup can be taken to be the orientation-preserving stabiliser of any hypergraph.

The proof of theorem 24 follows that of [OW09], replacing the theorems and lemmas used there with the versions stated above.

*Proof of theorem 15.* By theorem 24 random groups at density $d < \frac{1}{5}$ have a codimension-1 subgroup and it is proved in [NR98] that a group with a codimension-1 subgroup doesn't have Property (T). □

**Remark.** Although some of the lemmas in this paper have improved bounds, the bound in lemma 6.3 appears to be tight for this method, so the critical density is still $d < \frac{1}{5}$. As far as the author knows the best result known for the Gromov model, proved in [Mon21], is that random groups at density $d < \frac{3}{14}$ don't have Property (T).

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